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Painlevé I double scaling limit in the cubic matrix model

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Abstract

We obtain the double scaling asymptotic behavior of the recurrence coefficients and the partition function at the critical point of the cubic matrix model. We prove that the recurrence coefficients admit an asymptotic expansion in powers of $N^{-2/5}$, and in the leading order the asymptotic behavior of the recurrence coefficients is given by a Boutroux tronquée solution to the Painlevé I equation. We also obtain the double scaling limit of the partition function, and we prove that the poles of the tronquée solution are limits of zeros of the partition function. The tools used include the Riemann–Hilbert approach and the Deift–Zhou nonlinear steepest descent method for the corresponding family of complex orthogonal polynomials and their recurrence coefficients, together with the Toda equation in the parameter space.

Keywords : Random matrices, asymptotic representation in the complex domain, Riemann–Hilbert problems, topological expansion, partition function, double scaling limit, Painlevé I equation

AMS(MOS) Classification : Primary : 30E15, 60B20. Secondary : 33E17, 35Q15, 05C10.

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PAVEL BLEHER AND ALFREDO DEAÑO

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1. INTRODUCTION

We consider a unitary invariant ensemble of random matrices given by the probability distribution,

$$d\mu_N(M) = \frac{1}{\tilde{Z}_N} e^{-N\text{Tr}V(M)} dM, \quad (1.1)$$

where $V(M)$ is a polynomial,

$$V(M) = \sum_{k=1}^{2\nu} t_k M^k, \quad t_{2\nu} > 0. \quad (1.2)$$

Here M belongs to the space \mathcal{H}_N of $N \times N$ Hermitian matrices and dM is the Lebesgue measure on the space \mathcal{H}_N . The partition function of this model is equal to

$$\tilde{Z}_N = \int_{\mathbb{R}} \dots \int_{\mathbb{R}} e^{-N\text{Tr}V(M)} dM. \quad (1.3)$$

The probability distribution of eigenvalues is

$$d\mu_N(z_1, \dots, z_N) = \frac{1}{Z_N} \prod_{1 \leq j < k \leq N} (z_j - z_k)^2 \prod_{j=1}^N e^{-NV(z_j)} dz_1 \dots dz_N, \quad (1.4)$$

with the partition function

$$Z_N = \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \prod_{1 \leq j < k \leq N} (z_j - z_k)^2 \prod_{j=1}^N e^{-NV(z_j)} dz_1 \dots dz_N. \quad (1.5)$$

We define the free energy as

$$F_N := \frac{1}{N^2} \ln \frac{Z_N}{Z_N^0}, \quad (1.6)$$

where Z_N^0 corresponds to the Gaussian Unitary Ensemble (GUE), with $V(M) = M^2/2$. In this case the partition function (1.5) is a Selberg integral and can be computed explicitly, cf. [6]:

$$Z_N^0 = \frac{(2\pi)^{N/2}}{N^{N^2/2}} \prod_{n=1}^N n! \quad (1.7)$$

If we consider a deformation of the Gaussian model

$$V(M) = \frac{M^2}{2} + uW(M), \quad (1.8)$$

where u is a parameter, it is known that for small u the free energy admits an asymptotic expansion as $N \rightarrow \infty$,

$$F_N(u) \sim \sum_{g=0}^{\infty} \frac{F^{(2g)}(u)}{N^{2g}}, \quad N \rightarrow \infty, \quad (1.9)$$

which is called the *topological expansion*. This asymptotic expansion in inverse powers of N^2 for the free energy is well known in the literature. For unitary ensembles it can be obtained from the Riemann–Hilbert formulation, as presented by Ercolani and McLaughlin in [19], see also [20] for a more detailed result on the coefficients in the asymptotic expansion for potentials $V(M)$ with a dominant even power. Alternatively, it is possible to derive it via different techniques that are applicable to more general β -ensembles, see, for instance, the works of Borot and Guionnet [9] and [10].

It is a remarkable fact that the coefficients $F^{(2g)}(u)$ are generating functions for counting graphs on Riemannian surfaces of genus g . This important observation goes back to the earlier physical papers of Bessis, Itzykson and Zuber [5] and Brézin, Itzykson, Parisi and Zuber [12]. For rigorous mathematical proofs, see the works of Ercolani and McLaughlin [19], Ercolani, McLaughlin and Pierce [20], and the monograph of Forrester [23].

The case when $W(M)$ is a cubic polynomial is especially interesting in this direction, since it gives a generating function for counting triangulations on surfaces. For cubic $W(M)$, however, the partition function, as defined in (1.5), diverges, and it is necessary to consider its regularization. One way to achieve this is to consider integration on a specially chosen contour Γ in the complex plane:

$$Z_N(u) = \int_{\Gamma} \dots \int_{\Gamma} \prod_{1 \leq j < k \leq N} (z_j - z_k)^2 \prod_{j=1}^N e^{-N \left(\frac{z_j^2}{2} - u z_j^3 \right)} dz_1 \dots dz_N, \quad (1.10)$$

on which the integral converges. Strictly speaking, this partition function does not correspond to any Hermitian random matrix model, but it serves, nevertheless, as a generating function for triangulations on Riemannian surfaces, see, e.g. [7].

To choose an appropriate contour of integration Γ , consider the three sectors on the complex plane,

$$\begin{aligned} S_0 &= \left\{ z \in \mathbb{C} : \frac{5\pi}{6} < \arg z < \frac{7\pi}{6} \right\}, \\ S_1 &= \left\{ z \in \mathbb{C} : \frac{\pi}{6} < \arg z < \frac{\pi}{4} \right\}, \\ S_2 &= \left\{ z \in \mathbb{C} : -\frac{\pi}{4} < \arg z < -\frac{\pi}{6} \right\}, \end{aligned} \quad (1.11)$$

see Fig.1.

Then for any ray

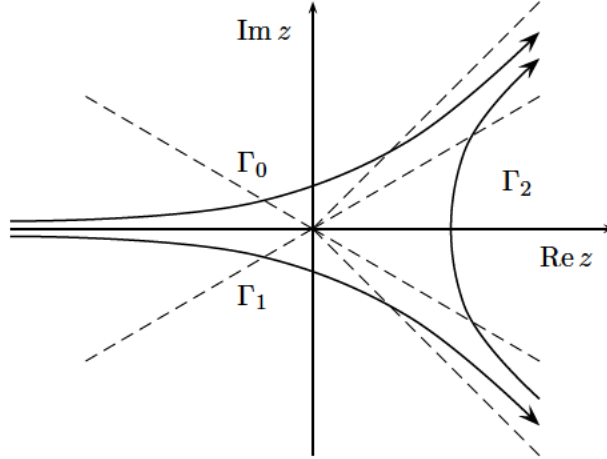
$$R_{\theta} = \{ z \in \mathbb{C} : \arg z = \theta \}, \quad (1.12)$$

lying in the sectors S_0 , S_1 , and S_2 , the integral

$$\int_{R_{\theta}} z^k e^{-N \left(\frac{z^2}{2} - u z^3 \right)} dz \quad (1.13)$$

converges for any $k = 0, 1, \dots$ and any $u \geq 0$. Clearly, it is also possible to take combinations of two such contours. In Fig. 1 we consider contours consisting of two rays joining the sectors S_0 , S_1 , and S_2 , namely

$$\Gamma_0 = R_{\pi} \cup R_{\pi/5}, \quad \Gamma_1 = R_{\pi} \cup R_{-\pi/5}, \quad \Gamma_2 = R_{-\pi/5} \cup R_{\pi/5}, \quad (1.14)$$


 FIGURE 1. The sectors S_0, S_1, S_2 and the contours $\Gamma_0, \Gamma_1, \Gamma_2$.

with orientation from $(-\infty)$ to $(\infty e^{\pi i/5})$ on Γ_0 , from $(-\infty)$ to $(\infty e^{-\pi i/5})$ on Γ_1 , and from $(\infty e^{-\pi i/5})$ to $(\infty e^{\pi i/5})$ on Γ_2 .

More generally, following [18] and also [3] and [21], it is convenient to introduce a linear combination of the contours Γ_0 and Γ_1 . To that end, let us fix some $\alpha \in \mathbb{C}$ and define

$$\Gamma = \alpha \Gamma_0 + (1 - \alpha) \Gamma_1, \quad (1.15)$$

in the sense that for any $f(z)$ such that the integral of $f(z)$ along Γ is well defined, we have

$$\int_{\Gamma} f(z) dz = \alpha \int_{\Gamma_0} f(z) dz + (1 - \alpha) \int_{\Gamma_1} f(z) dz. \quad (1.16)$$

With this choice of $\Gamma = \Gamma(\alpha)$, the integral $Z_N(u) = Z_N(u; \alpha)$ in (1.10) is convergent for any $u \geq 0$. By the Cauchy theorem, we have some flexibility in the choice of the contours Γ_0, Γ_1 within the sectors S_0, S_1, S_2 .

In our previous paper [7] we considered the large N asymptotic behavior of this free energy when u is in the interval $0 \leq u < u_c$, where u_c is the following critical value:

$$u_c = \frac{3^{1/4}}{18}. \quad (1.17)$$

In this regular regime, the free energy admits an asymptotic expansion in powers of N^{-2} of the form (1.9), see [7]. This expansion is uniform in the variable u on any interval $[0, u_c - \delta]$ for $\delta > 0$. Furthermore, the functions $F^{(2g)}(u)$ do not depend on α and they admit an analytic continuation to the disc $|u| < u_c$ in the complex plane.

The first two terms of the free energy can be written as power series in u , convergent for $|u| < u_c$, see [7]:

$$F^{(0)}(u) = \frac{1}{2} \sum_{j=1}^{\infty} \frac{72^j \Gamma\left(\frac{3j}{2}\right) u^{2j}}{\Gamma(j+3) \Gamma\left(\frac{j}{2} + 1\right)}, \quad (1.18)$$

and

$$F^{(2)}(u) = \frac{5}{48} \sum_{j=1}^{\infty} \frac{72^j \Gamma\left(\frac{3j}{2}\right)}{(3j+2) \Gamma(j+1) \Gamma\left(\frac{j}{2} + 1\right)} {}_3F_2\left(\begin{matrix} -j+1, 2, 6 \\ 5, -\frac{3j}{2} + 1 \end{matrix}; \frac{3}{2}\right) u^{2j}, \quad (1.19)$$

in terms of generalized hypergeometric functions, see [1, Chapter 16]. Consequently, we can write an asymptotic expansion for the partition function in the regular regime:

$$Z_N(u) = Z_N^0 e^{N^2 F^{(0)}(u) + F^{(2)}(u)} (1 + \mathcal{O}(N^{-2})), \quad N \rightarrow \infty. \quad (1.20)$$

In this paper we analyze the behavior of the free energy and the partition function near the critical case $u = u_c$. In this situation the equilibrium measure becomes singular, since its density vanishes with $3/2$ exponent at the right endpoint of the support. This corresponds to case III in the classification of singular equilibrium measures in [16]. Following [4, 18], where the authors work with models with quartic potential but similar local structure, a double scaling limit is needed, letting both $N \rightarrow \infty$ and $u \rightarrow u_c$ in a suitably coupled way. Then a different asymptotic expansion in powers of $N^{-2/5}$ is expected, involving solutions of the Painlevé I differential equation. We remark that this type of results are discussed in the articles by F. David, see [13] and [14].

The structure of the paper is as follows:

- In Section 2 we introduce the double scaling limit that we use near the critical point, and the family of (complex) orthogonal polynomials $P_n(z)$ that we will need in the analysis.
- In Section 3 we recall known properties of the Painlevé I differential equation, and we present a Riemann–Hilbert problem for an associated function Ψ . A uniform asymptotic expansion for this function is also proved in Theorem 3.2, a result that may be of independent interest.
- In Section 4 we present the main results of the paper: asymptotic expansions for the recurrence coefficients corresponding to the orthogonal polynomials $P_n(z)$ and for the free energy in the double scaling regime.
- In Section 5 we analyze the equilibrium measure and its support close to the critical case. Similarly as in [18], we need to construct a modified equilibrium measure, whose density becomes negative near one of the endpoints of the support.
- In Section 6 we apply the Deift–Zhou nonlinear steepest descent to the Riemann–Hilbert problem (see [17] or the monograph [15]), and as a result we deduce the asymptotic behavior of the recurrence coefficients $\gamma_{N,N}^2(u)$ and $\beta_{N,N}(u)$, corresponding to the orthogonal polynomials of degree $n = N$ with respect to the weight $e^{-NV(z;u)}$, in the double scaling regime and near the critical case. We also identify the subleading terms in these asymptotic expansions in terms of solutions of the Painlevé I differential equation, proving Theorem 4.1. Using the connection between the two double scaling formulations (in terms of $u - u_c$ and in terms of n/N , see (4.8)) we analyze the behavior of the recurrence coefficients $\gamma_{N,n}^2(u_c)$ and $\beta_{N,n}(u_c)$, corresponding to the orthogonal polynomials of degree n with respect to the weight function $e^{-NV(z;u_c)}$, at the critical case. This proves Corollary 4.2.
- In Section 8 we integrate the Toda equation to obtain the asymptotic behavior of the free energy near the critical case. To prove Theorem 4.6, we need to extend the regular and the double scaling regimes for the asymptotics of the recurrence coefficients. This implies a small modification of the corresponding local parametrices in the Riemann–Hilbert problem.

2. THE DOUBLE SCALING LIMIT

In the present paper we are interested in the large N asymptotic behavior of the free energy $F_N(u)$ in the double scaling regime, when $N \rightarrow \infty$ and $u \rightarrow u_c$ simultaneously, in such a way that

$$N^{4/5}(u - u_c) = c_1 \lambda, \quad (2.1)$$

where $c_1 > 0$ is a constant and $\lambda \in \mathbb{R}$ is a new variable. Note that the regular case $0 \leq u < u_c$ corresponds to $\lambda < 0$.

Similarly to [7], the analysis will make use of the monic polynomials $P_n(z) = z^n + \dots$ that are orthogonal in the following sense:

$$\int_{\Gamma} P_n(z) z^k w(z) dz = 0, \quad k = 0, 1, \dots, n-1, \quad (2.2)$$

where the path Γ is defined in (1.15) and the weight function is

$$w(z) = e^{-NV(z;u)}, \quad V(z;u) = \frac{z^2}{2} - uz^3. \quad (2.3)$$

The existence of such a sequence of orthogonal polynomials is not guaranteed a priori, since the weight function is not positive on Γ . However, one of the consequences of the Riemann–Hilbert analysis will be the existence of $P_n(z)$ for large enough N .

Following the work of Fokas, Its and Kitaev [21], we consider the following 2×2 Riemann–Hilbert problem (RHP) for the orthogonal polynomials $P_n(z)$:

Find a 2×2 matrix-valued $Y = Y_{n,N,u} : \mathbb{C} \setminus \Gamma \rightarrow \mathbb{C}^{2 \times 2}$ such that

- $Y(z)$ is analytic for $z \in \mathbb{C} \setminus \Gamma$ and the limits

$$\lim_{s \rightarrow z} Y(s) = Y_{\pm}(z) \quad (2.4)$$

exist as s approaches z from the \pm -side of Γ . As usual, we assume that the $+$ side is on the left of an oriented contour, and the $-$ side is on the right.

- For $s \in \Gamma$, the boundary values $Y_{\pm}(s)$ are related by a jump matrix:

$$Y_+(s) = Y_-(s) \begin{cases} \begin{pmatrix} 1 & \alpha w(s) \\ 0 & 1 \end{pmatrix}, & s \in \Gamma_0, \\ \begin{pmatrix} 1 & (1-\alpha)w(s) \\ 0 & 1 \end{pmatrix}, & s \in \Gamma_1. \end{cases} \quad (2.5)$$

- As $z \rightarrow \infty$,

$$Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}. \quad (2.6)$$

We call n the *degree* of the RHP. This RHP has a unique solution if and only if the monic polynomial $P_n(z)$, orthogonal with respect to the weight function $w(z)$ uniquely exists. If additionally $P_{n-1}(z)$ uniquely exists, then the solution of the RHP is given by:

$$Y(z) = Y_n(z) = \begin{pmatrix} P_n(z) & (\mathcal{C}P_n w)(z) \\ -\frac{2\pi i}{h_{n-1}} P_{n-1}(z) & -\frac{2\pi i}{h_{n-1}} (\mathcal{C}P_{n-1} w)(z) \end{pmatrix}, \quad (2.7)$$

where

$$(\mathcal{C}f)(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(s)}{s-z} ds \quad (2.8)$$

is the Cauchy transform of f on Γ , and the coefficient h_{n-1} is defined as

$$h_{n-1} = \int_{\Gamma} P_{n-1}^2(s) w(s) ds. \quad (2.9)$$

Conversely, we have the following result:

Proposition 2.1. *Suppose that the RHP (2.4)-(2.6) has a solution $Y_n(z)$ for some n . Then the orthogonal polynomial $P_n(z)$ exists uniquely.*

Proof. The proof is standard using the jump matrix and orthogonality, see for instance [15]. \square

A very important identity satisfied by the sequence of orthogonal polynomials, if it exists, is the following three term recurrence relation:

Proposition 2.2. *Suppose that RHP (2.4)-(2.6) has solutions $Y_{n-1}(z)$, $Y_n(z)$, and $Y_{n+1}(z)$ for the degrees $n-1$, n , and $n+1$, respectively. Then the orthogonal polynomials $P_{n-1}(z)$, $P_n(z)$, and $P_{n+1}(z)$, which uniquely exist by Proposition 2.1, satisfy the three term recurrence relation,*

$$zP_n(z) = P_{n+1}(z) + \beta_n P_n(z) + \gamma_n^2 P_{n-1}(z). \quad (2.10)$$

Proof. This result is standard in the theory of orthogonal polynomials. We can write

$$zP_n(z) = c_0 P_{n+1}(z) + c_1 P_n(z) + c_2 P_{n-1}(z) + \sum_{k=3}^{n+1} c_k z^{n-k+1}, \quad (2.11)$$

with some suitable coefficients. Then multiplication by monomials z^j for $j = 0, 1, \dots, n-2$ and integration on Γ , together with the orthogonality property (2.2), gives the result. \square

Note that both the orthogonal polynomials and the recurrence coefficients in the above relation depend on the parameter u , and also on n and N . When needed, we will denote these recurrence coefficients by $\gamma_{N,n}^2(u)$ and $\beta_{N,n}(u)$, to stress this dependence.

3. RIEMANN–HILBERT PROBLEM FOR THE PAINLEVÉ I EQUATION

3.1. The Painlevé I equation. In order to state our results, we will need to work with solutions of the Painlevé I differential equation. This is a nonlinear second-order ordinary differential equation that in standard form reads

$$y''(\lambda) = 6y(\lambda)^2 + \lambda, \quad (3.1)$$

see for instance [1, Chapter 32]. Because of the Painlevé property, it is known, from the original work of Painlevé [27, §17–§19], that any solution of this differential equation is a meromorphic function in \mathbb{C} with infinitely many double poles. Furthermore, if we denote the set of these poles by \mathcal{P} , then in a vicinity of any pole $\lambda_j \in \mathcal{P}$, the function $y(\lambda)$ can be expanded in Laurent series:

$$y(\lambda) = \frac{1}{(\lambda - \lambda_j)^2} - \frac{\lambda_j}{10}(\lambda - \lambda_j)^2 - \frac{1}{6}(\lambda - \lambda_j)^3 + C(\lambda - \lambda_j)^4 + \mathcal{O}((\lambda - \lambda_j)^5), \quad (3.2)$$

where C is an arbitrary constant.

We also note the following \mathbb{Z}_5 -symmetry, that follows directly from the differential equation: if $y(\lambda)$ is a solution of (3.1), then the function

$$y^0(\lambda) = \frac{1}{\omega_5^2} y\left(\frac{\lambda}{\omega_5}\right), \quad \omega_5 = e^{2\pi i/5}, \quad (3.3)$$

is a solution as well. So we consider the rays

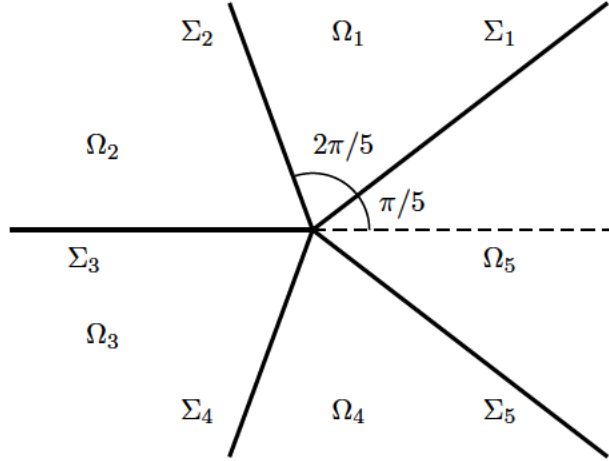
$$\Sigma_k = \left\{ \lambda \in \mathbb{C} : \arg \lambda = \frac{\pi}{5} + \frac{2(k-1)\pi}{5}, k = 1, 2, \dots, 5 \right\}, \quad (3.4)$$

that delimit the sectors

$$\Omega_k = \left\{ \lambda \in \mathbb{C} : \frac{\pi}{5} + \frac{2(k-1)\pi}{5} < \arg \lambda < \frac{\pi}{5} + \frac{2k\pi}{5}, k = 1, 2, \dots, 5 \right\}, \quad (3.5)$$

see Figure 2.

These canonical sectors were originally considered by Boutroux [11] in the analysis of solutions to Painlevé I. As proved in [11], for any $\lambda_j \in \mathbb{C}$, not located on any of the rays indicated in Figure 2, there exists a solution $y(\lambda)$ to Painlevé I that has a double pole at the point λ_j and that is *tronquée*


 FIGURE 2. Boutroux sectors in \mathbb{C} for the solutions of Painlevé I.

in the direction of the semi axis Σ_3 , in the following sense: for any $\varepsilon > 0$ there exists $R > 0$ such that in the region

$$\Lambda = \{\lambda \in \mathbb{C} : |\lambda| \geq R, \frac{3\pi}{5} + \varepsilon < \arg \lambda < \frac{7\pi}{5} - \varepsilon\} \quad (3.6)$$

the function $y(\lambda)$ is free of poles. A similar result holds in the other sectors because of the \mathbb{Z}_5 symmetry shown before.

Moreover, the solution has the following asymptotic behavior in Λ :

$$y(\lambda) \sim \sqrt{-\frac{\lambda}{6}} \sum_{k=0}^{\infty} a_k (-\lambda)^{-5k/2}, \quad |\lambda| \rightarrow \infty. \quad (3.7)$$

Here the coefficients a_k are given by the nonlinear recursion

$$a_0 = 1, \quad a_{k+1} = \frac{25k^2 - 1}{8\sqrt{6}} a_k - \frac{1}{2} \sum_{m=1}^k a_m a_{k+1-m}, \quad k \geq 0, \quad (3.8)$$

so the first ones are

$$a_0 = 1, \quad a_1 = -\frac{1}{8\sqrt{6}}, \quad a_2 = -\frac{49}{768}. \quad (3.9)$$

3.2. The Ψ function and its Riemann–Hilbert problem. Kapaev in [25] describes tronquée solutions of Painlevé I in terms of the following Riemann–Hilbert problem: introduce a 2×2 matrix valued function $\Psi(\zeta; \lambda, \alpha)$ such that

- (1) Ψ is analytic on $\mathbb{C} \setminus \Gamma_\Psi$, where

$$\Gamma_\Psi = \gamma_1 \cup \gamma_2 \cup \rho \cup \gamma_{-2} \cup \gamma_{-1} \quad (3.10)$$

is the contour shown in Figure 3.

- (2) On the contour Γ_Ψ , oriented as in Figure 3, the positive (from the left) and negative (from the right) limit values of $\Psi(\zeta)$, denoted $\Psi_\pm(\zeta)$ respectively, satisfy

$$\Psi_+ = \Psi_- J_\Psi \quad (3.11)$$

where the jump matrices are indicated in Figure 3:

$$J_1 = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \quad J_{-1} = \begin{pmatrix} 1 & 1-\alpha \\ 0 & 1 \end{pmatrix}, \quad J_2 = J_{-2} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad J_\rho = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3.12)$$

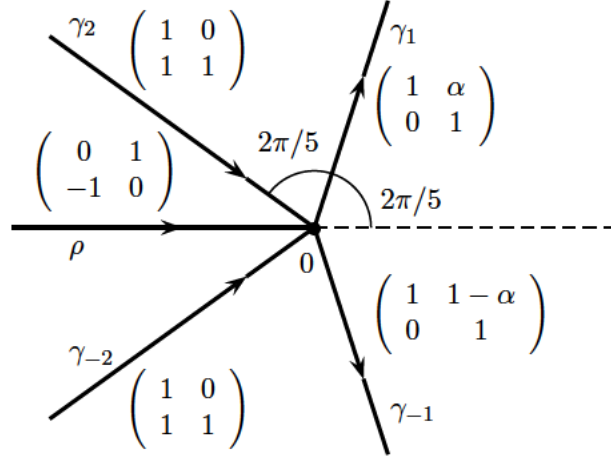


FIGURE 3. The contour Γ_Ψ and the jumps for the function Ψ associated with the Painlevé I differential equation.

(3) As $\zeta \rightarrow \infty$, with fixed λ , we have the asymptotic series

$$\Psi(\zeta; \lambda, \alpha) = \frac{\zeta^{\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \left(I + \frac{\Psi_1(\lambda, \alpha)}{\zeta^{1/2}} + \frac{\Psi_2(\lambda, \alpha)}{\zeta} + \mathcal{O}(\zeta^{-3/2}) \right) e^{\theta(\zeta; \lambda) \sigma_3}. \quad (3.13)$$

where $\Psi_1(\lambda, \alpha)$ is a diagonal matrix and the phase function is

$$\theta(\zeta; \lambda) = \frac{4}{5} \zeta^{5/2} + \lambda \zeta^{1/2}, \quad (3.14)$$

with fractional powers taken on the principal sheet and a cut on the negative half-axis. We use the standard notation for the Pauli matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.15)$$

In [25] it is proved that this Riemann–Hilbert problem has a unique solution for large enough $|\lambda|$ in the sector Λ . Furthermore, the function $\Psi(\zeta; \lambda, \alpha)$ satisfies the following system of linear ODEs:

$$\begin{aligned} \Psi_\zeta \Psi^{-1} &= \begin{pmatrix} -(y_\alpha)_\lambda & 2\zeta^2 + 2y_\alpha \zeta + \lambda + 2y_\alpha^2 \\ 2\zeta - 2y_\alpha & (y_\alpha)_\lambda \end{pmatrix} \\ \Psi_\lambda \Psi^{-1} &= \begin{pmatrix} 0 & \zeta + 2y_\alpha \\ 1 & 0 \end{pmatrix}. \end{aligned} \quad (3.16)$$

The compatibility condition of this Lax pair implies the Painlevé I equation (3.1) for the function $y_\alpha(\lambda)$. Here the subscripts $(\cdot)_\lambda$ and $(\cdot)_\zeta$ indicate differentiation with respect to these variables.

The parameter α that appears in the Riemann–Hilbert problem parametrizes a family of tronquée solutions of Painlevé I, with asymptotic behavior

$$y_\alpha(\lambda) \sim \sqrt{-\frac{\lambda}{6}} \sum_{k=0}^{\infty} a_k (-\lambda)^{-5k/2}, \quad (3.17)$$

as $|\lambda| \rightarrow \infty$ in Λ . The dependence on α appears in exponentially small corrections to this asymptotic behavior. These extra terms are computed by Kapaev in [25, Theorem 2.2]. If $\arg \lambda \in [3\pi/5, \pi]$,

then the large $|\lambda|$ asymptotics of $y_\alpha(\lambda)$ is given by

$$y_\alpha(\lambda) = y_1(\lambda) + \frac{i(1-\alpha)}{\sqrt{\pi} 2^{11/8} (-3\lambda)^{1/8}} e^{-\frac{2^{11/4} 3^{1/4}}{5} (-\lambda)^{5/4}} \left(1 + \mathcal{O}(\lambda^{-3/8})\right), \quad |\lambda| \rightarrow \infty, \quad (3.18)$$

where $y_1(\lambda)$ behaves as (3.17).

A similar result holds in the sector $\arg \lambda \in [\pi, 7\pi/5]$, with respect to $y_0(\lambda)$ instead of $y_1(\lambda)$. As a consequence, [25, Corollary 2.4] establishes that if $\arg \lambda \in [3\pi/5, 7\pi/5]$, then

$$y_0(\lambda) - y_1(\lambda) = \frac{i}{\sqrt{\pi} 2^{11/8} (-3\lambda)^{1/8}} e^{-\frac{2^{11/4} 3^{1/4}}{5} (-\lambda)^{5/4}} \left(1 + \mathcal{O}(\lambda^{-3/8})\right), \quad |\lambda| \rightarrow \infty. \quad (3.19)$$

Furthermore, the cases $\alpha = 0$ and $\alpha = 1$ are special in the sense that, as $|\lambda| \rightarrow \infty$, we have

$$\begin{aligned} y_0(\lambda) &= \sqrt{-\frac{\lambda}{6}} + \mathcal{O}(\lambda^{-2}), & \varepsilon + \frac{3\pi}{5} < \arg \lambda < \frac{11\pi}{5} - \varepsilon, \\ y_1(\lambda) &= \sqrt{-\frac{\lambda}{6}} + \mathcal{O}(\lambda^{-2}), & \varepsilon - \frac{\pi}{5} < \arg \lambda < \frac{7\pi}{5} - \varepsilon, \end{aligned} \quad (3.20)$$

for arbitrary $\varepsilon > 0$. This means that these are *tritonquée* solutions (asymptotically free of poles in four out of the five canonical sectors), again after Boutroux [11], see also [24].

The first two terms $\Psi_1(\lambda, \alpha)$ and $\Psi_2(\lambda, \alpha)$ in the asymptotic expansion (3.13) are fixed and can be written as follows, cf. [18]:

$$\begin{aligned} \Psi_1(\lambda, \alpha) &= -\mathcal{H}_\alpha(\lambda) \sigma_3 = \begin{pmatrix} -\mathcal{H}_\alpha(\lambda) & 0 \\ 0 & \mathcal{H}_\alpha(\lambda) \end{pmatrix}, \\ \Psi_2(\lambda, \alpha) &= \frac{1}{2} \mathcal{H}_\alpha(\lambda)^2 I + \frac{1}{2} y_\alpha(\lambda) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathcal{H}_\alpha(\lambda)^2 & -iy_\alpha(\lambda) \\ iy_\alpha(\lambda) & \mathcal{H}_\alpha(\lambda)^2 \end{pmatrix}. \end{aligned} \quad (3.21)$$

Here

$$\mathcal{H}_\alpha(\lambda) = \frac{1}{2} (y'_\alpha(\lambda))^2 - 2y_\alpha^3(\lambda) - y_\alpha(\lambda)\lambda \quad (3.22)$$

is the Hamiltonian corresponding to Painlevé I. Consequently, given a fixed value of α , the function $y_\alpha(\lambda)$ can be written in terms of the solution of this Riemann–Hilbert problem:

$$y_\alpha(\lambda) = 2i(\Psi_2(\lambda))_{12}. \quad (3.23)$$

Remark 3.1. *The parameter α , that appears again in the jumps of the Riemann–Hilbert problem, can be naturally related to the Stokes multipliers for Painlevé I, as formulated by Kapaev, namely $\alpha = -is_1$ or $\alpha = 1 + is_{-1}$. Note however that the function $\Psi(\zeta; \lambda, \alpha)$ that we just defined is not exactly the same that is used in [25]. If we denote this last one by $\Psi^{(0)}$, we have the relation*

$$\Psi(\zeta; \lambda, \alpha) = \Psi^{(0)}(\zeta; \lambda, \alpha) \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}, \quad (3.24)$$

and the jump matrices are related as follows:

$$J_\Psi = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} J_{\Psi^{(0)}} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}. \quad (3.25)$$

3.3. Large $|\lambda|$ asymptotics for the solution of the Painlevé I Riemann–Hilbert problem.

The asymptotic expansion (3.13) holds for large ζ and fixed λ , but for the analysis of the free energy later on, we need to extend it to cover the case when $(-\lambda) \rightarrow \infty$, $\lambda \in \mathbb{R}$, simultaneously. To this end, we define the following function:

$$\Phi(\zeta; \lambda, \alpha) = (-\lambda)^{-\sigma_3/8} \Psi(\zeta(-\lambda)^{1/2}; \lambda, \alpha). \quad (3.26)$$

Then Φ solves the following RH problem:

- (1) Φ is analytic on $\mathbb{C} \setminus \Gamma_\Psi$.
- (2) On Γ_Ψ , Φ has the jumps,

$$\Phi_+ = \Phi_- J_\Psi. \quad (3.27)$$

- (3) As $\zeta \rightarrow \infty$, Φ expands in the asymptotic series

$$\Phi(\zeta; \lambda, \alpha) \sim \frac{\zeta^{\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \left(I + \sum_{k=1}^{\infty} \frac{\Psi_k(\lambda, \alpha)}{(-\lambda)^{k/4} \zeta^{k/2}} \right) e^{(-\lambda)^{5/4} \theta_0(\zeta) \sigma_3}, \quad (3.28)$$

where now

$$\theta_0(\zeta) = \frac{4}{5} \zeta^{5/2} - \zeta^{1/2}. \quad (3.29)$$

The first equation in (3.16) gives the differential equation for Φ :

$$\Phi_\zeta \Phi^{-1} = (-\lambda)^{5/4} A \quad (3.30)$$

where

$$A = A(\zeta; \lambda, \alpha) = \begin{pmatrix} -(y_\alpha)_\lambda (-\lambda)^{-3/4} & 2\zeta^2 + 2y_\alpha \zeta (-\lambda)^{-1/2} - 1 + 2(y_\alpha)^2 (-\lambda)^{-1} \\ 2\zeta - 2y_\alpha (-\lambda)^{-1/2} & (y_\alpha)_\lambda (-\lambda)^{-3/4} \end{pmatrix}, \quad (3.31)$$

where again $(\cdot)_\lambda$ indicates differentiation with respect to this variable.

It is possible to derive a semiclassical solution to this system for large $|\lambda|$ by using the asymptotic behavior of $y_\alpha(\lambda)$. As $\lambda \rightarrow -\infty$, the matrix A has the following limit:

$$\begin{aligned} \lim_{\lambda \rightarrow -\infty} A(z; \lambda, \alpha) &= A_\infty(\zeta) = \begin{pmatrix} 0 & 2\zeta^2 + \frac{2\zeta}{\sqrt{6}} - \frac{2}{3} \\ 2(\zeta - \frac{1}{\sqrt{6}}) & 0 \end{pmatrix} \\ &= 2 \begin{pmatrix} \zeta - \frac{1}{\sqrt{6}} \\ 1 \end{pmatrix} \begin{pmatrix} 0 & \zeta + \frac{2}{\sqrt{6}} \\ 1 & 0 \end{pmatrix}. \end{aligned} \quad (3.32)$$

The eigenvalues of A_∞ are

$$\zeta_{1,2} = \pm 2 \left(\zeta - \frac{1}{\sqrt{6}} \right) \left(\zeta + \frac{2}{\sqrt{6}} \right)^{1/2} \quad (3.33)$$

and we introduce the g -function as

$$g(\zeta) = \frac{4}{5} \left(\zeta + \frac{2}{\sqrt{6}} \right)^{5/2} - \frac{2\sqrt{6}}{3} \left(\zeta + \frac{2}{\sqrt{6}} \right)^{3/2}, \quad (3.34)$$

in such a way that

$$g'(\zeta) = 2 \left(\zeta - \frac{1}{\sqrt{6}} \right) \left(\zeta + \frac{2}{\sqrt{6}} \right)^{1/2}. \quad (3.35)$$

As $\zeta \rightarrow \infty$, $g(\zeta)$ has the following power series expansion:

$$g(\zeta) = \frac{4}{5} \zeta^{5/2} - \zeta^{1/2} - \frac{\sqrt{6}}{9} \zeta^{-1/2} + \frac{1}{24} \zeta^{-3/2} - \frac{\sqrt{6}}{180} \zeta^{-5/2} + \dots \quad (3.36)$$

Hence, the first two terms in the expansion coincide with the phase function $\theta_0(\lambda; \alpha)$ in (3.29), and the subsequent terms are small as $\zeta \rightarrow \infty$, so we can rewrite the asymptotic expansion (3.28) as follows:

$$\Phi(\zeta; \lambda, \alpha) \sim \frac{\zeta^{\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \left(I + \sum_{k=1}^{\infty} \frac{\Phi_k(\lambda, \alpha)}{(-\lambda)^{k/4} \zeta^{k/2}} \right) e^{(-\lambda)^{5/4} g(\zeta) \sigma_3}, \quad (3.37)$$

with some modified coefficients $\Phi_k(\lambda, \alpha)$.

Regarding a large $|\lambda|$ asymptotic expansion for the function $\Phi(\zeta; \lambda, \alpha)$, uniform in ζ , we have the following result:

Theorem 3.2. Fix $\alpha \in \mathbb{C}$, then $\Phi(\zeta; \lambda, \alpha)$ has the following asymptotic behavior as $(-\lambda) \rightarrow \infty$:

$$\Phi(\zeta; \lambda, \alpha) = \left(I + \mathcal{O} \left(\frac{1}{(-\lambda)^{5/4}(1+|\zeta|)} \right) \right) \frac{(\zeta - \zeta_0)^{\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} e^{(-\lambda)^{5/4}g(\zeta)\sigma_3}, \quad (3.38)$$

where

$$\zeta_0 = -\frac{2}{\sqrt{6}}, \quad (3.39)$$

and $g(\zeta)$ is given by (3.34). The estimate holds uniformly for $\zeta \in \mathbb{C} \setminus D(\zeta_0, \varepsilon)$, for any $\varepsilon > 0$.

In the proof of this result we use the Riemann–Hilbert approach to analyze $\Phi(\zeta; \lambda, \alpha)$, and we consider the sequence of transformations of the Deift–Zhou steepest descent method. The proof can be found in the appendix.

4. STATEMENT OF MAIN RESULTS

We prove the following results:

Theorem 4.1. Fix $\alpha \in \mathbb{C}$, and suppose that $\lambda \in \mathbb{R}$ is not a pole of the Painlevé I function $y_\alpha(\lambda)$. Under the double scaling relation

$$N^{4/5}(u - u_c) = c_1\lambda, \quad c_1 = 2^{-12/5}3^{-7/4}, \quad (4.1)$$

the recurrence coefficients for the polynomials orthogonal with respect to the weight $e^{-NV(z;u)}$, with $V(z;u) = \frac{z^2}{2} - uz^3$, satisfy

$$\begin{aligned} \gamma_{N,N}^2(u) &\sim \gamma_c^2 + \sum_{k=1}^{\infty} \frac{1}{N^{2k/5}} p_{2k}(\lambda), \\ \beta_{N,N}(u) &\sim \beta_c + \sum_{k=1}^{\infty} \frac{1}{N^{2k/5}} q_{2k}(\tilde{\lambda}). \end{aligned} \quad (4.2)$$

for some coefficients $p_{2k}(\lambda)$ and $q_{2k}(\tilde{\lambda})$. The leading terms are

$$\gamma_c^2 = \sqrt{3}, \quad \beta_c = 3^{1/4}(\sqrt{3} - 1), \quad (4.3)$$

and $\tilde{\lambda}$ is a shifted variable defined in terms of λ as

$$\tilde{\lambda} = N^{4/5}\varphi \left(\varphi^{-1}(\lambda N^{-4/5}) + \frac{1}{2N} \right), \quad \varphi(t) = \frac{\sqrt{1+t}-1}{c_2}(1+t)^{4/5}. \quad (4.4)$$

Furthermore, we have

$$p_2(\lambda) = -2^{4/5}3^{1/2}y_\alpha(\lambda), \quad (4.5)$$

in terms of the solution to Painlevé I, and $q_2(\lambda) = 3^{-1/4}p_2(\lambda)$.

We can prove a similar result with the double scaling in terms of n/N . Note that if we apply the change of variables

$$z = \frac{u}{u_c}\xi, \quad (4.6)$$

then we obtain $NV(z; u_c) = nV(\xi; u)$, where $n = Nu^2/u_c^2$. This leads to an alternative double scaling for the ratio n/N :

$$\frac{n}{N} = \frac{u^2}{u_c^2} = (1 + c_2\lambda N^{-4/5})^2, \quad (4.7)$$

where $c_2 = c_1/u_c = 2^{-7/5}$ is a new constant. Consequently, we define a new variable v as follows:

$$\frac{n}{N} = 1 + vN^{-4/5}. \quad (4.8)$$

In this setting, we fix $u = u_c$ and we let both $n, N \rightarrow \infty$, in such a way that v in the previous formula remains fixed. We have the following result:

Corollary 4.2. *Fix $u = u_c$, and let n and N satisfy the double scaling relation (4.8), with $v \in \mathbb{R}$ bounded. Then the recurrence coefficients satisfy*

$$\begin{aligned}\gamma_{N,n}^2 &\sim \gamma_c^2 + \sum_{k=1}^{\infty} \frac{1}{N^{2k/5}} \hat{p}_{2k}(v), \\ \beta_{N,n} &\sim \beta_c + \sum_{k=1}^{\infty} \frac{1}{N^{2k/5}} \hat{q}_{2k}(\tilde{v}),\end{aligned}\tag{4.9}$$

for some coefficients $\hat{p}_{2k}(v)$ and $\hat{q}_{2k}(\tilde{v})$, and with the same values of γ_c^2 and β_c as in the previous theorem. The variable v is defined in (4.8) and \tilde{v} is a shifted variable defined in terms of v as

$$\tilde{v} = v + \frac{N^{-1/5}}{2},\tag{4.10}$$

in such a way that

$$\frac{n}{N} + \frac{1}{2N} = 1 + \tilde{v}n^{-4/5}.\tag{4.11}$$

Remark 4.3. *As noticed in [18], the terms of order $N^{-1/5}$ cancel out in the asymptotic expansion of the recurrence coefficients. Using the shifted variables $\tilde{\lambda}$ and \tilde{v} before, this is true in general for higher order odd terms.*

In order to formulate our main result on the asymptotic behavior of the partition function, we need to extend the asymptotic expansion of the recurrence coefficients both from the regular and the double scaling regime, in order to be able to integrate the Toda equation. Namely, we have the following result:

Theorem 4.4. *The asymptotic expansions (4.2) for $\gamma_{N,N}^2(u)$ and $\beta_{N,N}(u)$ hold for $N^{4/5}(u - u_c) = c_1\lambda$, where $\lambda = \mathcal{O}(N^{4/25-\varepsilon_1})$, for $\varepsilon_1 > 0$. Simultaneously, for any $K \geq 1$ and $u - u_c = \mathcal{O}(N^{-4/5+\varepsilon_2})$, for $\varepsilon_2 > 0$, we have the truncated regular expansion*

$$\begin{aligned}\gamma_{N,N}^2(u) &= g_0(u) + \sum_{k=1}^K \frac{g_{2k}(u)}{N^{2k}} + \mathcal{O}(N^{-\frac{2}{5}-\frac{5K+4}{2}\varepsilon_2}), \\ \beta_{N,N}(u) &= b_0(u) + \sum_{k=1}^K \frac{b_{2k}(u)}{N^{2k}} + \mathcal{O}(N^{-\frac{2}{5}-\frac{5K+4}{2}\varepsilon_2}),\end{aligned}\tag{4.12}$$

with the coefficients $g_{2k}(u)$ and $b_{2k}(u)$ from the regular regime, see [7].

Remark 4.5. *The estimate of the error term comes from the extended regular regime $u - u_c = \mathcal{O}(N^{-4/5+\varepsilon_2})$, together with [7], where it is shown that*

$$g_{2k}(u), b_{2k}(u) = \mathcal{O}((u_c - u)^{\frac{1}{2}-\frac{5k}{2}}),\tag{4.13}$$

for $k \geq 1$. The zeroth terms satisfy

$$g_0(u) = \sqrt{3} + \mathcal{O}((u_c - u)^{\frac{1}{2}}), \quad b_0(u) = 3^{1/4}(\sqrt{3} - 1) + \mathcal{O}((u_c - u)^{\frac{1}{2}}).\tag{4.14}$$

As a consequence of this theorem, in the intermediate region $\lambda = \mathcal{O}(N^{\varepsilon_3})$, with $0 < \varepsilon_3 < 4/25$, we can match both asymptotic expansions and integrate to obtain the behavior of the free energy.

Using this information, we can prove our main result on the asymptotic behavior of the partition function:

Theorem 4.6. *Given $\varepsilon > 0$ and $\delta > 0$, consider the double scaling regime (4.1), $N^{4/5}(u - u_c) = c_1 \lambda$ and fix a neighborhood in the complex plane $D_R = \{\lambda \in \mathbb{C} : |\lambda| < R\}$. Let $\{\lambda_{\alpha,j}\}_{j=1}^J$ be the set of poles of $y_\alpha(\lambda)$ in D_R . The partition function $Z_N(u)$ can be written in the following way:*

$$Z_N(u) = Z_N^{\text{reg}}(u) Z_N^{\text{sing}}(\lambda) (1 + \mathcal{O}(N^{-\varepsilon})), \quad (4.15)$$

for $\lambda \in D_R \setminus \cup_j D(\lambda_j, \delta)$. Here the regular part is

$$Z_N^{\text{reg}}(u) = e^{N^2[A+B(u-u_c)+C(u-u_c)^2]+D}, \quad (4.16)$$

where the constants A , B , C and D are explicit:

$$A = F^{(0)}(u_c), \quad B = F^{(0)'}(u) \Big|_{u=u_c}, \quad C = \frac{1}{2} F^{(0)''}(u) \Big|_{u=u_c}, \quad (4.17)$$

where $F^{(0)}(u)$ is given by (1.18), and

$$D = \left[F^{(2)}(u) + \frac{1}{48} \ln(u_c - u) \right] \Big|_{u=u_c}, \quad (4.18)$$

where $F^{(2)}(u)$ is given by (1.19). The singular part of the partition function is

$$Z_N^{\text{sing}}(\lambda) = e^{-Y_\alpha(\lambda)}, \quad (4.19)$$

where $Y_\alpha(\lambda)$ solves the differential equation

$$Y_\alpha''(\lambda) = y_\alpha(\lambda), \quad (4.20)$$

with boundary condition

$$Y_\alpha(\lambda) = \frac{2\sqrt{6}}{45}(-\lambda)^{5/2} - \frac{1}{48} \log(-\lambda), \quad (-\lambda) \rightarrow \infty. \quad (4.21)$$

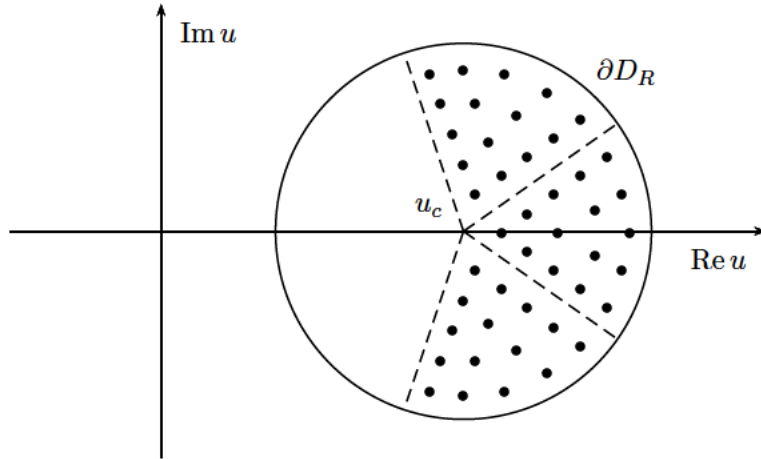


FIGURE 4. Illustration of the domain used in Theorem 4.6. Dots represent poles of $y_\alpha(\lambda)$, schematically. Here $\partial D_R = \{u \in \mathbb{C} : N^{4/5}|u - u_c| = c_1 R\}$.

As a consequence we can relate the poles of $y_\alpha(\lambda)$ with zeros of the partition function, cf. [14]:

Corollary 4.7. *Assume that there are no poles of $y_\alpha(\lambda)$ on the boundary of the disc ∂D_R , then for large N , the partition function $Z_N(u)$ has exactly J zeros $\tau_j(N)$ in D_R , and $\tau_j(N) \rightarrow \lambda_j$ as $N \rightarrow \infty$.*

Proof. Taking into account the behavior of $y_\alpha(\lambda)$ near the poles, see (3.2), we obtain by integration

$$Y_\alpha(\lambda) = -\log(\lambda - \lambda_j) + \mathcal{O}((\lambda - \lambda_j)^4), \quad j = 1, 2, \dots, J, \quad (4.22)$$

so on the boundary of $D(\lambda_j, \delta)$ we have

$$Z_N^{\text{sing}}(\lambda) = (\lambda - \lambda_j)(1 + \mathcal{O}(\lambda - \lambda_j)). \quad (4.23)$$

By the argument principle, $Z_N^{\text{sing}}(\lambda)$ has exactly one zero in $D(\lambda_j, \delta)$. \square

In Figure 4 we illustrate the setting of the theorem schematically. Note that the variable used is u instead of λ , so the boundary of the disc is the set $\{u \in \mathbb{C} : N^{4/5}(u - u_c) = c_1\lambda\}$, and it will shrink with N for fixed λ .

5. THE EQUILIBRIUM MEASURE

As shown in [19], a key element in the analysis of the partition function and free energy is the equilibrium measure in the external field $V(z; u)$. In the cubic case this equilibrium measure can be written explicitly, as we show next.

5.1. Support of the equilibrium measure. When $0 \leq u < u_c$, we know from results in [7] that the equilibrium measure is supported on an interval $[a, b]$ of the real axis, and the density is

$$\varrho_u(z) = \frac{1}{2\pi} \sqrt{(z-a)(b-z)}(1-3uz-3ux), \quad (5.1)$$

where both a and b depend on u , and $x = (a+b)/2$.

The parameter $\tau = ux$ satisfies the following cubic equation, see [7]:

$$18\tau^3 - 9\tau^2 + \tau - 6u^2 = 0. \quad (5.2)$$

Denote $s = 108\sqrt{3}u^2$, then the cubic equation becomes

$$18\tau^3 - 9\tau^2 + \tau - \frac{\sqrt{3}}{54}s = 0, \quad (5.3)$$

and it has three solutions, that behave as follows as $s \rightarrow 0$:

$$\tau_1(s) = \frac{1}{3} + \frac{\sqrt{3}}{54}s + \mathcal{O}(s^2), \quad \tau_2(s) = \frac{1}{6} - \frac{\sqrt{3}}{27}s + \mathcal{O}(s^2), \quad \tau_3(s) = \frac{\sqrt{3}}{54}s + \frac{1}{108}s^2 + \mathcal{O}(s^3). \quad (5.4)$$

Furthermore, the discriminant of the cubic equation (5.3) is $\Delta(s) = 9(1-s^2)$. When $s = 1$, the cubic equation (5.3) has a double root and a single root:

$$\tau_1(1) = \frac{1}{6} + \frac{\sqrt{3}}{9}, \quad \tau_2(1) = \tau_3(1) = \frac{1}{6} - \frac{\sqrt{3}}{18}, \quad (5.5)$$

and similarly when $s = -1$:

$$\tau_1(-1) = \tau_2(-1) = \frac{1}{6} + \frac{\sqrt{3}}{18}, \quad \tau_3(-1) = \frac{1}{6} - \frac{\sqrt{3}}{9}. \quad (5.6)$$

We are interested in the solution $x(u)$ that is bounded near $u = 0$, so for the analysis we choose the solution $\tau_3(s)$.

For convenience, we make the following linear change of variables (depending on u):

$$\zeta = \frac{2z - a - b}{b - a}, \quad (5.7)$$

so that the interval $[a, b]$ is mapped to $[-1, 1]$. Then the equilibrium measure becomes

$$\varrho_u(\zeta)d\zeta = -\frac{3(b-a)^3u}{16\pi}\sqrt{1-\zeta^2}\left(\zeta - \frac{1-6ux}{3uy}\right)d\zeta, \quad (5.8)$$

where $y = (b-a)/2$. This parameter satisfies the equation

$$y^2 = \frac{4}{1-6ux}, \quad (5.9)$$

so

$$y = \frac{2}{\sqrt{1-6ux}} = \frac{2}{\sqrt{1-6\tau}}, \quad (5.10)$$

since we assume that $b > a$ and therefore $y > 0$.

The extra root of $\varrho_u(\zeta)$ is

$$\zeta_0 = \frac{1-6ux}{3uy} = \frac{1}{6u}(1-6\tau)^{3/2}, \quad (5.11)$$

which is outside the interval $[-1, 1]$ when $u < u_c$ (actually $\zeta_0 > 1$), and coalesces with the right endpoint when $u = u_c$, see [7].

Near the critical case $s = 1$, we write $\Delta s = 1 - s$ and by perturbation we obtain

$$\tau_{3\pm}(s) = \frac{1}{6} - \frac{\sqrt{3}}{18} \pm \frac{\sqrt{2}}{18}(\Delta s)^{1/2} + \frac{\sqrt{3}}{162}\Delta s + \mathcal{O}((\Delta s)^{3/2}). \quad (5.12)$$

For y , we have the solutions

$$y_3(s) = 2 \cdot 3^{1/4} + 2^{1/2}3^{-1/4}(\Delta s)^{1/2} + \frac{11 \cdot 3^{1/4}}{18}\Delta s + \mathcal{O}((\Delta s)^{3/2}). \quad (5.13)$$

Regarding the double root, we have

$$\zeta_0 = 1 + \frac{\sqrt{6}}{2}(\Delta s)^{1/2} + \frac{7}{12}\Delta s + \mathcal{O}((\Delta s)^{3/2}), \quad (5.14)$$

taking the solution $\tau_{3+}(s)$.

Note that this is consistent with the expected behavior: if $u < u_c$ then $s < 1$, and $\zeta_0 > 1$, so the double root is outside the interval where the equilibrium measure is supported. If $u > u_c$ then $s > 1$, and both the endpoints and the double root become complex. A more complete picture of the different cases that can occur in this cubic model, using the notion of S -curves in the complex plane and numerical computations, is described in [2, §4]. Note also that the dependence of the endpoints on the parameter s is not analytic near the critical case.

5.2. Modified equilibrium measure at the critical case. In the new variable ζ , the polynomial $V(z; u) = \frac{z^2}{2} - uz^3$ becomes

$$V(\zeta; u) = V_{cr}(\zeta) + (u - u_c)V^o(\zeta). \quad (5.15)$$

where

$$V_{cr}(\zeta) = \frac{(2\zeta + \sqrt{3} - 1)^2(-2\zeta + 2\sqrt{3} + 1)}{6} \quad (5.16)$$

and

$$V^o(\zeta) = -3^{3/4}(2\zeta + \sqrt{3} - 1)^3. \quad (5.17)$$

The equilibrium measure at the critical time has the density, from (5.8):

$$\varrho_{cr}(x)dx = \frac{2}{\pi}(1-x)\sqrt{1-x^2}dx, \quad -1 \leq x \leq 1. \quad (5.18)$$

We observe that this is indeed a probability density function, but the equilibrium measure is not regular, since its density vanishes with a $3/2$ exponent at the right endpoint. In this respect, the

analysis is very similar to the one carried out in [18], but without the symmetry around the origin present in that case.

The resolvent of $\varrho_{cr}(x)$ is

$$\omega_{cr}(\zeta) = \int_{-1}^1 \frac{\varrho_{cr}(x)dx}{\zeta - x}, \quad \zeta \in \mathbb{C} \setminus [-1, 1] \quad (5.19)$$

and it solves the equation

$$\omega_{cr+}(x) + \omega_{cr-}(x) = V'_{cr}(x) = -4x^2 + 4x + 2, \quad x \in (-1, 1). \quad (5.20)$$

It is equal to

$$\omega_{cr}(\zeta) = \frac{V'_{cr}(\zeta)}{2} + \pi i \varrho_{cr}(\zeta) = -2\zeta^2 + 2\zeta + 1 + 2(\zeta + 1)^{1/2}(\zeta - 1)^{3/2}. \quad (5.21)$$

Let us extend the function $\varrho_{cr}(x)$ to the complex plane as

$$\varrho_{cr}(\zeta) = \frac{2i}{\pi}(\zeta + 1)^{1/2}(\zeta - 1)^{3/2}, \quad \zeta \in \mathbb{C} \setminus [-1, 1], \quad (5.22)$$

with a cut on $[-1, 1]$, so that

$$\varrho_{cr}(x + i0) = \frac{2}{\pi}(x + 1)^{1/2}(1 - x)^{3/2}, \quad x \in (-1, 1). \quad (5.23)$$

In what follows the following function will be important for us:

$$\phi_{cr}(\zeta) = -2\pi i \int_1^\zeta \varrho_{cr}(s)ds = 4 \int_1^\zeta (s + 1)^{1/2}(s - 1)^{3/2}ds, \quad \zeta \in \mathbb{C} \setminus (-\infty, 1], \quad (5.24)$$

where the integration goes over the segment $[1, \zeta]$ on the complex plane. It can be integrated explicitly as

$$\phi_{cr}(\zeta) = \frac{2\sqrt{\zeta^2 - 1}(\zeta - 2)(2\zeta + 1)}{3} + 2 \log \left(\zeta + \sqrt{\zeta^2 - 1} \right). \quad (5.25)$$

We plot the level curves of the function $\operatorname{Re} \phi_{cr}(\zeta)$ in Figure 5. From this illustration, it seems reasonable to take the following contour Γ_Y for the Riemann–Hilbert analysis: the real axis from $-\infty$ to $\zeta = 1$ and then a combination of steepest descent path of $\operatorname{Re} \phi_{cr}(\zeta)$ into the upper and lower half plane, on which $\operatorname{Im} \phi_{cr}(\zeta) = 0$, see Figure 5. This combination is determined by the arbitrary complex parameter α .

5.3. Modified equilibrium measure near the critical case. Following the ideas in [18, §4.2], but without the symmetry present in the quartic case, we construct a modified equilibrium measure $\mu_u(x)$ in order to study the problem near the critical case. This measure solves a minimization problem over signed measures supported on the interval $[\sigma_u, 1]$. The left endpoint σ_u will depend on u , and will become -1 when $u = u_c$.

Because of the Euler–Lagrange equations for the equilibrium measure μ_u , see for instance [15], we have

$$2 \int \log \frac{1}{|x - y|} d\mu_u(y) + V(x; u) + \ell_u = 0, \quad x \in (\sigma_u, 1), \quad (5.26)$$

for some constant ℓ_u . Differentiating with respect to x , we obtain

$$2 \cdot \operatorname{PV} \int \frac{d\mu_u(y)}{x - y} = V'(x; u), \quad x \in (\sigma_u, 1). \quad (5.27)$$

Differentiating (5.15), we get

$$V'(\zeta; u) = V'_{cr}(\zeta) + (u - u_c)V''_{cr}(\zeta) = -4\zeta^2 + 4\zeta + 2 - 2 \cdot 3^{7/4} \left(2\zeta - 1 + \sqrt{3} \right)^2 (u - u_c), \quad (5.28)$$

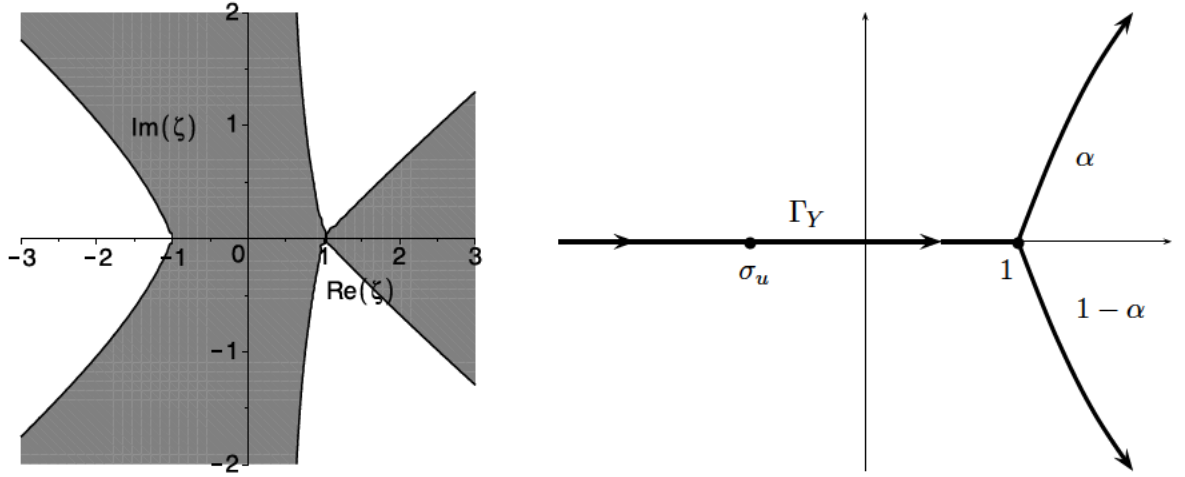


FIGURE 5. On the left, level curves of $\operatorname{Re} \phi_{cr}(\zeta)$ in the critical case $u = u_c$. In white, the region where $\operatorname{Re} \phi_{cr}(\zeta) < 0$, in grey, the region where $\operatorname{Re} \phi_{cr}(\zeta) > 0$. On the right, the contour Γ_Y for the Riemann-Hilbert formulation.

using (5.7) again. This modified equilibrium measure must verify

$$\begin{aligned} \int_{\sigma_u}^1 d\mu_u(y) &= 1, \\ \operatorname{PV} \int_{\sigma_u}^1 \frac{d\mu_u(y)}{x-y} &= -2x^2 + 2x + 1 - 3^{7/4} (2x - 1 + \sqrt{3})^2 (u - u_c), \quad x \in (\sigma_u, 1). \end{aligned} \quad (5.29)$$

The resolvent,

$$\omega_u(\zeta) = \int_{\sigma_u}^1 \frac{d\mu_u(y)}{\zeta - y}, \quad \zeta \in \mathbb{C} \setminus [\sigma_u, 1], \quad (5.30)$$

because of (5.29), must satisfy the following:

$$\omega_u(\zeta) = \frac{1}{\zeta} + \mathcal{O}(\zeta^{-2}), \quad \zeta \rightarrow \infty, \quad (5.31)$$

$$\omega_{u+}(x) + \omega_{u-}(x) = -4x^2 + 4x + 2 - 2 \cdot 3^{7/4} (2x - 1 + \sqrt{3})^2 (u - u_c), \quad x \in (\sigma_u, 1).$$

Consequently, we look for $\omega_u(\zeta)$ in the form

$$\omega_u(\zeta) = -2\zeta^2 + 2\zeta + 1 - 3^{7/4} (2\zeta - 1 + \sqrt{3})^2 (u - u_c) - \frac{m_u(\zeta)\sqrt{\zeta - \sigma_u}}{2\sqrt{\zeta - 1}}, \quad \zeta \in \mathbb{C} \setminus [\sigma_u, 1], \quad (5.32)$$

where $\sqrt{\zeta - \sigma_u}/\sqrt{\zeta - 1}$ is taken on the principal sheet, with a cut on $[\sigma_u, 1]$. From the second equation in (5.31), it follows that $m_u(\zeta)$ has no jump on $(\sigma_u, 1)$, and hence it should be analytic in \mathbb{C} . From the first equation in (5.31), we obtain that $m_u(\zeta)$ is a quadratic polynomial:

$$m_u(\zeta) = a_2(\zeta - 1)^2 + a_1(\zeta - 1) + a_0. \quad (5.33)$$

Imposing the first condition in (5.31), we can derive equations for the coefficients a_0 , a_1 and a_2 :

$$\begin{aligned} a_2 &= -4 - 8 \cdot 3^{7/4}(u - u_c), \\ a_1 &= -2(\sigma_u + 1) - 12 \cdot 3^{3/4}(\sigma_u + 2\sqrt{3} + 1)(u - u_c) \\ a_0 &= -\frac{1}{2}(\sigma_u + 1)(3\sigma_u - 5) - 3^{7/4}(3\sigma_u^2 + (4\sqrt{3} - 2)\sigma_u + 7)(u - u_c). \end{aligned} \quad (5.34)$$

The endpoint σ_u is a solution of the following cubic equation:

$$-\frac{1}{8}(\sigma_u + 1)(5\sigma_u^2 - 14\sigma_u + 13) - \frac{3^{7/4}}{4}(\sigma_u - 1)(5\sigma_u^2 + (6\sqrt{3} - 4)\sigma_u + 7 - 2\sqrt{3})(u - u_c) = 0. \quad (5.35)$$

This last equation can be solved by perturbation, and we obtain

$$\sigma_u = -1 + 3^{7/4}(2 - \sqrt{3})(u - u_c) + \mathcal{O}((u - u_c)^2). \quad (5.36)$$

Furthermore, the discriminant of the cubic equation (5.35) is

$$\Delta_u = -16 - 12 \cdot 3^{1/4}(14\sqrt{3} + 9)(u - u_c) + \mathcal{O}((u - u_c)^2), \quad (5.37)$$

so for small $|u - u_c|$ we have $\Delta_u \neq 0$, and therefore we have three different solutions. In particular, the solution σ_u is analytic in u in a neighborhood of $u = u_c$.

Using (5.36), we can write the coefficients a_0 , a_1 and a_2 in powers of $u - u_c$:

$$\begin{aligned} a_2 &= -4 - 8 \cdot 3^{7/4}(u - u_c), \\ a_1 &= 8 + 18 \cdot 3^{3/4}(2 - \sqrt{3})(u - u_c) + \mathcal{O}((u - u_c)^2), \\ a_0 &= -4 + 2 \cdot 3^{5/4}(9 - 4\sqrt{3})(u - u_c) + \mathcal{O}((u - u_c)^2). \end{aligned} \quad (5.38)$$

Higher order terms in $u - u_c$ can be computed by the same procedure. Note that when $u = u_c$ we have $m_{u_c}(\zeta) = -4\zeta + 8\zeta - 4 = -4(\zeta - 1)^2$ and $\sigma_{u_c} = -1$, so

$$\omega_{u_c}(\zeta) = -2\zeta^2 + 2\zeta + 1 + 2(\zeta + 1)^{1/2}(\zeta - 1)^{3/2} \quad \zeta \in \mathbb{C} \setminus [-1, 1], \quad (5.39)$$

which recovers (5.21). Finally, if we write the density as $d\mu_u(\zeta) = \psi_u(\zeta)d\zeta$, then

$$\psi_u(x) = -\frac{1}{2\pi i}(\omega_{u+}(x) - \omega_{u-}(x)) = -\frac{m_u(x)\sqrt{x - \sigma_u}}{2\pi\sqrt{1 - x}}, \quad x \in (\sigma_u, 1). \quad (5.40)$$

6. THE RIEMANN-HILBERT PROBLEM

Consider the contour Γ_Y on Figure 5 and the following Riemann-Hilbert problem (RHP) for a 2×2 matrix-valued function $Y = Y_{N,n,u} : \mathbb{C} \setminus \Gamma_Y \rightarrow \mathbb{C}^{2 \times 2}$:

(1) $Y(\zeta)$ is analytic in $\mathbb{C} \setminus \Gamma_Y$, and for every $s \in \Gamma_Y$ the limits

$$Y_{\pm}(s) = \lim_{\zeta \rightarrow s, \zeta \in \Omega_{\pm}} Y(\zeta), \quad (6.1)$$

exist, where Ω_{\pm} are the left and the right sides of Γ_Y , respectively, oriented as in Figure 5.

(2) On Γ_Y , the function $Y(\zeta)$ has a multiplicative jump:

$$Y_+(s) = Y_-(s) \begin{cases} \begin{pmatrix} 1 & e^{-NV(s;u)} \\ 0 & 1 \end{pmatrix}, & s \in (-\infty, 1] \\ \begin{pmatrix} 1 & \alpha e^{-NV(s;u)} \\ 0 & 1 \end{pmatrix}, & s \in \Gamma_Y \cap \{\operatorname{Im} s > 0\} \\ \begin{pmatrix} 1 & (1 - \alpha)e^{-NV(s;u)} \\ 0 & 1 \end{pmatrix}, & s \in \Gamma_Y \cap \{\operatorname{Im} s < 0\}. \end{cases} \quad (6.2)$$

(3) As $\zeta \rightarrow \infty$,

$$Y(\zeta) = (I + \mathcal{O}(\zeta^{-1})) \begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix}. \quad (6.3)$$

We note that all the matrices in the subsequent transformations will depend on u , n and N , but we will omit this for brevity.

6.1. The g -function. The g -function associated with the modified equilibrium measure $d\mu_u(x)$ is

$$g_u(\zeta) = \int_{\sigma_u}^1 \log(\zeta - y) d\mu_u(y). \quad (6.4)$$

It has the following properties:

- (1) $g_u(\zeta)$ is analytic for $\zeta \in \mathbb{C} \setminus (-\infty, 1]$, and it has limiting values $g_{u\pm}(x)$ as $\zeta \rightarrow x \pm i0$, $x \in (-\infty, 1)$.
- (2) For $\zeta \in \mathbb{C} \setminus (-\infty, 1]$,

$$\frac{dg_u(\zeta)}{d\zeta} = \omega_u(\zeta). \quad (6.5)$$

- (3) By (5.26), $g_u(x)$ satisfies the Euler-Lagrange equation,

$$g_{u+}(x) + g_{u-}(x) - V(x; u) - \ell_u = 0, \quad x \in (\sigma_u, 1), \quad (6.6)$$

- (4) As $\zeta \rightarrow \infty$,

$$g_u(\zeta) = \log \zeta + \mathcal{O}(\zeta^{-1}). \quad (6.7)$$

- (5) By (6.4), the difference of boundary values of $g(x)$ on the real axis is

$$G_u(x) \equiv g_{u+}(x) - g_{u-}(x) = \begin{cases} 0, & x \in [1, \infty) \\ 2\pi i \int_x^1 d\mu_u(y), & x \in (\sigma_u, 1) \\ 2\pi i, & x \in (-\infty, \sigma_u]. \end{cases} \quad (6.8)$$

- (6) Since the density of $d\mu_u(x)$, $\psi_u(x)$, is analytic on $(\sigma_u, 1)$, the function $G_u(x)$ is analytic on $(\sigma_u, 1)$ too, and by (6.8) and the Cauchy–Riemann equations,

$$\left. \frac{G_u(x + iy)}{dy} \right|_{y=0} = 2\pi\psi_u(x), \quad \sigma_u < x < 1. \quad (6.9)$$

In what follows, the function

$$\phi_u(\zeta) = 2g_u(\zeta) - V(\zeta; u) - \ell_u, \quad \zeta \in \mathbb{C} \setminus (-\infty, 1], \quad (6.10)$$

will be important. By adding equations (6.6) and (6.8), we obtain that

$$\phi_{u+}(x) = 2g_{u+}(x) - V(x; u) - \ell_u = 2\pi i \int_x^1 \psi_u(y) dy, \quad x \in (\sigma_u, 1). \quad (6.11)$$

and also $\phi_{u+}(x) = G_u(x)$ for $x \in (\sigma_u, 1)$. By analytic continuation, we can extend this equation from $x \in (\sigma_u, 1)$ to $\zeta \in \mathbb{C} \setminus (-\infty, 1]$. To this end we first extend the density $\psi_u(x)$, for $x \in (\sigma_u, 1)$, to a function $r_u(\zeta)$ analytic on $\mathbb{C} \setminus [\sigma_u, 1]$ such that

$$r_{u\pm}(x) = \pm\psi_u(x), \quad x \in (\sigma_u, 1). \quad (6.12)$$

Equation (6.11) can be now analytically extended to $\zeta \in \mathbb{C} \setminus (-\infty, 1]$ as

$$\phi_u(\zeta) = 2g_u(\zeta) - V_u(\zeta) - \ell_u = 2\pi i \int_{\zeta}^1 r_u(s) ds, \quad \zeta \in \mathbb{C} \setminus (-\infty, 1], \quad (6.13)$$

where integration is taken over the segment joining ζ and 1 in the complex plane.

Observe that similar to (6.11),

$$\phi_{u-}(x) = 2g_u(x) - V_u(x) - \ell_u = -2\pi i \int_x^1 \psi_u(y) dy, \quad x \in (\sigma_u, 1), \quad (6.14)$$

and $\phi_{u-}(x) = -G_u(x)$ for $x \in (\sigma_u, 1)$. Also,

$$\phi_{u\pm}(x) = \pm 2\pi i + 2\pi i \int_x^{\sigma_u} r_u(y) dy, \quad x \in (-\infty, \sigma_u]. \quad (6.15)$$

The function $G_u(x)$, defined in (6.8), is analytic on the interval $(\sigma_u, 1)$. By (6.11) and (6.14) it is analytically extended to the set $\mathbb{C} \setminus ((-\infty, \sigma_u] \cup [1, \infty))$, and

$$\phi_u(\zeta) = \pm G_u(\zeta) \quad \text{for} \quad \pm \operatorname{Im} \zeta > 0. \quad (6.16)$$

6.2. First transformation of the RHP. Define $T(\zeta)$ as

$$T(\zeta) = e^{-\frac{N\ell_u\sigma_3}{2}} Y(\zeta) e^{-N[g_u(\zeta) - \frac{\ell_u}{2}]\sigma_3}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (6.17)$$

Then $T(\zeta)$ solves the following RHP:

(1) $T(\zeta)$ is analytic in $\mathbb{C} \setminus \Gamma_Y$, and for every $s \in \Gamma_Y$ the following limits exist:

$$T_{\pm}(s) = \lim_{\zeta \rightarrow s, \zeta \in \Omega_{\pm}} T(\zeta). \quad (6.18)$$

(2) The jumps for the matrix T are as follows:

$$T_+(s) = T_-(s) \begin{cases} \begin{pmatrix} e^{-NG_u(s)} & 0 \\ 0 & e^{NG_u(s)} \end{pmatrix}, & s \in (\sigma_u, 1) \\ \begin{pmatrix} 1 & \alpha e^{N\phi_u(s)} \\ 0 & 1 \end{pmatrix}, & s \in \Gamma_Y \cap \{\operatorname{Im} s > 0\} \\ \begin{pmatrix} 1 & (1-\alpha)e^{N\phi_u(s)} \\ 0 & 1 \end{pmatrix}, & s \in \Gamma_Y \cap \{\operatorname{Im} s < 0\} \\ \begin{pmatrix} 1 & e^{N\tilde{\phi}_u(s)} \\ 0 & 1 \end{pmatrix}, & s \in (-\infty, \sigma_u], \end{cases} \quad (6.19)$$

where

$$\tilde{\phi}_u(x) = 2\pi i \int_x^{\sigma_u} r_u(y) dy, \quad x \in (-\infty, \sigma_u], \quad (6.20)$$

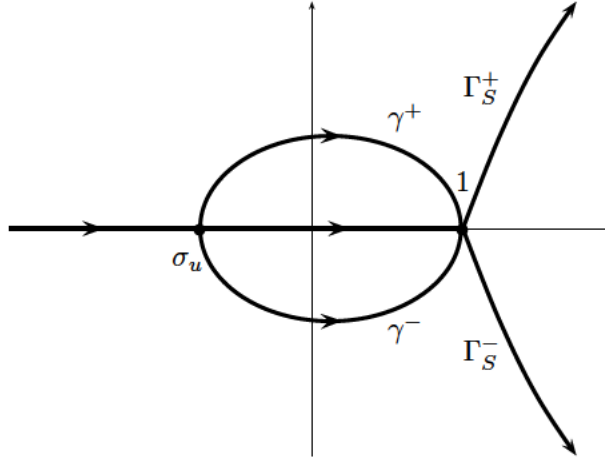
using (6.11), (6.14) and (6.15).

(3) As $\zeta \rightarrow \infty$,

$$T(\zeta) = I + \mathcal{O}(\zeta^{-1}). \quad (6.21)$$

6.3. Second transformation of the RHP: Opening of lenses. The jump matrix $J_T(x; u)$ can be factored as follows on the interval $(\sigma_u, 1)$:

$$\begin{pmatrix} e^{-NG_u(x)} & 1 \\ 0 & e^{NG_u(x)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{NG_u(x)} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-NG_u(x)} & 1 \end{pmatrix}. \quad (6.22)$$


 FIGURE 6. The contour Γ_S for the Riemann-Hilbert analysis.

We introduce the contour Γ_S as shown on Figure 6, and we set

$$S(\zeta) = \begin{cases} T(\zeta) \begin{pmatrix} 1 & 0 \\ -e^{-NG_u(\zeta)} & 1 \end{pmatrix}, & \text{if } \zeta \text{ is in the upper part of the lens,} \\ T(\zeta) \begin{pmatrix} 1 & 0 \\ e^{NG_u(\zeta)} & 1 \end{pmatrix}, & \text{if } \zeta \text{ is in the lower part of the lens,} \\ T(\zeta), & \text{otherwise.} \end{cases} \quad (6.23)$$

Then $S(\zeta)$ solves the following RHP:

- (1) $S(\zeta)$ is analytic in $\mathbb{C} \setminus \Gamma_S$, and for every $s \in \Gamma_S$ the following limits exist:

$$S_{\pm}(s) = \lim_{\zeta \rightarrow s, \zeta \in \Omega_{\pm}} S(\zeta). \quad (6.24)$$

- (2) The matrix S has the following jumps:

$$S_+(s) = S_-(s) \begin{cases} \begin{pmatrix} 1 & e^{N\tilde{\phi}_u(s)} \\ 0 & 1 \end{pmatrix}, & s \in (-\infty, \sigma_u], \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & s \in (\sigma_u, 1), \\ \begin{pmatrix} 1 & \alpha e^{N\phi_u(s)} \\ 0 & 1 \end{pmatrix}, & s \in \Gamma_S^+ \\ \begin{pmatrix} 1 & (1-\alpha)e^{N\phi_u(s)} \\ 0 & 1 \end{pmatrix}, & s \in \Gamma_S^- \\ \begin{pmatrix} 1 & 0 \\ e^{-NG_u(s)} & 1 \end{pmatrix}, & s \in \gamma^+, \\ \begin{pmatrix} 1 & 0 \\ e^{NG_u(s)} & 1 \end{pmatrix}, & s \in \gamma^-, \end{cases} \quad (6.25)$$

where γ^+ and γ^- are the curved boundaries of the upper and lower part of the lens, respectively. Recall that the function $\tilde{\phi}_u(s)$ is given by (6.20).

(3) As $\zeta \rightarrow \infty$,

$$S(\zeta) = I + \mathcal{O}(\zeta^{-1}). \quad (6.26)$$

Note that because of (6.16), we can rewrite the jump conditions on the lips of the lens as follows:

$$J_S(s; u) = \begin{pmatrix} 1 & 0 \\ e^{-N\phi_u(s)} & 1 \end{pmatrix}, \quad s \in \gamma^+ \cup \gamma^-. \quad (6.27)$$

As observed in [18], the use of the modified equilibrium measure causes some difficulties when opening the lens. If the density of the modified measure $\psi_u(y) > 0$, then (6.8) implies

$$\frac{d}{dx} \operatorname{Im} G_u(x + 0i) = -2\pi i \psi_u(x) < 0, \quad x \in (\sigma_u, 1), \quad (6.28)$$

and then, because of the Cauchy–Riemann equations, we deduce that

$$\frac{d}{dy} \operatorname{Re} G_u(x + iy) > 0, \quad x \in (\sigma_u, 1), \quad (6.29)$$

and the off-diagonal entries of the jump matrix $J_S(s; u)$ on the lips of the lens $\gamma_+ \cup \gamma_-$, see (6.25), are exponentially decaying. However, the modified density $\psi_u(y)$ becomes negative for x close to 1, so we obtain exponentially increasing terms. This can be controlled if $|u - u_c|$ is small enough, though, as shown in the following lemmas:

Lemma 6.1. *Let U be a neighborhood of the point 1, given $\varepsilon > 0$ there exists $\delta > 0$ such that for every $u \in \mathbb{R}$ with $|u - u_c| < \delta$, we have $\operatorname{Re} \phi_u > \varepsilon$ on the lips of the lens outside U .*

Proof. The point where the density $\psi_u(x)$ changes sign in $[\sigma_u, 1]$ can be computed explicitly to leading order in $u - u_c$, using (5.40). In fact, we have two roots of the factor $m_u(\zeta)$:

$$x_{\pm} = 1 \pm 3^{7/8} \sqrt{5 - 2\sqrt{3}} (u_c - u)^{1/2} + \mathcal{O}(u - u_c), \quad (6.30)$$

and so given $\varepsilon > 0$, we can find $\delta > 0$ small enough such that for $|u - u_c| < \delta$, the function ψ_u is positive in the interval (σ_u, x_-) , and then $\operatorname{Re} \phi_u > \varepsilon$ on the lips of the lens, away from a small neighborhood of the endpoint $x = 1$. \square

On the contours Γ_S^{\pm} , we can prove the following:

Lemma 6.2. *For any $\varepsilon > 0$ there exists $\delta > 0$ in such a way that if $|u - u_c| < \delta$ then on the curve*

$$\Gamma_S^{\pm\varepsilon} = \Gamma_S \cap \{\pm \operatorname{Im} s \geq \varepsilon\}$$

there is a constant $C > 0$ such that we have the inequality,

$$\phi_u(s) \leq -C|s - 1|^3, \quad s \in \Gamma_Y^{\varepsilon}. \quad (6.31)$$

Proof. We use the fact that

$$\phi_u(\zeta) = \phi_{cr}(\zeta) + (u - u_c)\phi^o(\zeta), \quad (6.32)$$

together with the explicit expression for $\phi_{cr}(\zeta)$ given in (5.25). In particular, we have

$$\phi_{cr}(\zeta) = \frac{4}{3}\zeta^3 + \mathcal{O}(\zeta^2), \quad \zeta \rightarrow \infty, \quad (6.33)$$

so if we substitute $\zeta = 1 + re^{\pm\pi i/5}$, we have

$$\phi_{cr}(r) = Cr^3 + \mathcal{O}(r^2), \quad \operatorname{Re} C < 0. \quad (6.34)$$

A similar estimate can be computed for the term $\phi^o(\zeta)$:

$$\phi^o(r) = \tilde{C}_u r^3 + \mathcal{O}(r^2), \quad \operatorname{Re} C_u < 0, \quad (6.35)$$

if $|u - u_c|$ is small enough, and then we have the desired decay. \square

Lemma 6.2 implies that for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|u - u_c| \leq \delta$ then

$$J_S(s; u) = \begin{pmatrix} 1 & \mathcal{O}(e^{-c(s)N}) \\ 0 & 1 \end{pmatrix}, \quad s \in \Gamma_S^\varepsilon, \quad (6.36)$$

where $c(s) = C|s - 1|^3$, with $C > 0$.

6.4. Model solution. The model solution $M(\zeta)$ solves the following RHP:

- (1) $M(\zeta)$ is analytic in $\mathbb{C} \setminus [\sigma_u, 1]$, and for every $x \in (\sigma_u, 1)$ the following limits exist:

$$M_\pm(s) = \lim_{\zeta \rightarrow s, \zeta \in \Omega_\pm} M(\zeta), \quad (6.37)$$

- (2) On $[\sigma_u, 1]$,

$$M_+(x) = M_-(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (6.38)$$

- (3) As $\zeta \rightarrow \infty$,

$$M(\zeta) = I + \mathcal{O}(\zeta^{-1}). \quad (6.39)$$

The model solution is explicitly equal to

$$M(\zeta) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \beta^{-1}(\zeta) & 0 \\ 0 & \beta(\zeta) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\beta(\zeta) + \beta^{-1}(\zeta)}{2} & \frac{\beta(\zeta) - \beta^{-1}(\zeta)}{-2i} \\ \frac{\beta(\zeta) - \beta^{-1}(\zeta)}{2i} & \frac{\beta(\zeta) + \beta^{-1}(\zeta)}{2} \end{pmatrix}, \quad (6.40)$$

where

$$\beta(\zeta) = \left(\frac{\zeta - \sigma_u}{\zeta - 1} \right)^{1/4}, \quad (6.41)$$

with a cut on $[\sigma_u, 1]$ and the branch such that $\beta(\infty) = 1$.

6.5. Parametrix around the point $\zeta = 1$. The local parametrix near the endpoint $\zeta = 1$ will be given in terms of a certain solution of the Painlevé I differential equation. We consider a disc $D(1, \varepsilon) = \{\zeta : |\zeta - 1| \leq \varepsilon\}$ around the right endpoint of the support. We seek a function $P(\zeta)$ that satisfies the following Riemann–Hilbert problem:

- (1) $P(\zeta)$ is analytic in $\mathbb{C} \setminus \Gamma_S$, see Figure 6, and for every $s \in \Gamma_S$ the limits

$$P_\pm(s) = \lim_{\zeta \rightarrow s, \zeta \in \Omega_\pm} P(\zeta), \quad (6.42)$$

exist.

- (2) On $D(1, \varepsilon) \cap \Gamma_S$, the function P has the following jumps:

$$P_+(s) = P_-(s) \begin{cases} \begin{pmatrix} 1 & \alpha e^{N\phi_u(s)} \\ 0 & 1 \end{pmatrix}, & s \in D(1, \varepsilon) \cap \Gamma_S^+, \\ \begin{pmatrix} 1 & (1 - \alpha)e^{N\phi_u(s)} \\ 0 & 1 \end{pmatrix}, & s \in D(1, \varepsilon) \cap \Gamma_S^-, \\ \begin{pmatrix} 1 & 0 \\ e^{-N\phi_u(s)} & 1 \end{pmatrix}, & s \in D(1, \varepsilon) \cap (\gamma^+ \cup \gamma^-), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & s \in D(1, \varepsilon) \cap (\sigma_u, 1). \end{cases} \quad (6.43)$$

- (3) As $\zeta \rightarrow \infty$,

$$P(\zeta) = M(\zeta) \left(I + \mathcal{O}(N^{-1/5}) \right), \quad (6.44)$$

uniformly for $\zeta \in \partial D(1, \varepsilon)$.

We observe that if we consider the function

$$\tilde{P}(\zeta) = P(\zeta)e^{N\phi_u(\zeta)\sigma_3/2}. \quad (6.45)$$

in $D(1, \varepsilon)$, then by (6.25), this new function has the following jumps in $\Gamma_{\tilde{P}} = \Gamma_S \cap D(1, \varepsilon)$:

$$\tilde{P}_+(s) = \tilde{P}_-(s) \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & s \in [1 - \varepsilon, 1), \\ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, & s \in \Gamma_S^+ \cap D(1, \varepsilon), \\ \begin{pmatrix} 1 & 1 - \alpha \\ 0 & 1 \end{pmatrix}, & s \in \Gamma_S^- \cap D(1, \varepsilon), \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & s \in \gamma^\pm. \end{cases} \quad (6.46)$$

To see this, we recall that $\phi_{u\pm}(s) = \pm G_u(s)$ for $s \in (\sigma_u, 1)$ by formula (6.11) and (6.14), and also $\phi_u(s) = \pm G_u(s)$ for $s \in \gamma^\pm$ because of (6.16).

The jumps $J_{\tilde{P}}(s; u)$ fit well to the ones of the $\Psi(\zeta; \lambda, \alpha)$ -functions given in Section 3.2. In the spirit of [18], we look for a local parametrix in a neighborhood of the endpoint $\zeta = 1$ using this function:

Lemma 6.1. *Let $P : D(1, \varepsilon) \setminus \Sigma_S \mapsto \mathbb{C}^{2 \times 2}$ be defined by the formula*

$$P(\zeta) = E(\zeta)\Psi(N^{2/5}f(\zeta); N^{4/5}h_u(\zeta), \alpha)e^{-N\phi_u(\zeta)\sigma_3/2}, \quad (6.47)$$

where

- the function $\Psi(w; \lambda, \alpha)$ is the solution of the Riemann–Hilbert problem stated in Section 3.2.
- $f(\zeta)$ is a conformal map from $D(1, \varepsilon)$ to a neighborhood of 0 such that $\Gamma_S \cap D(1, \varepsilon)$ (see Figure 7) is mapped onto part of the contour in Figure 3. Explicitly,

$$f(\zeta) = \left[\frac{5}{8}\phi_{cr}(\zeta) \right]^{2/5}. \quad (6.48)$$

- $h_u : D(1, \varepsilon) \mapsto \mathbb{C}$ is analytic and such that $N^{4/5}h_u(D(1, \varepsilon))$ does not contain any poles of the solution $y_\alpha(\lambda)$ of Painlevé I. Explicitly,

$$h_u(\zeta) = \left(\frac{1}{20} \right)^{1/5} \frac{\phi_u(\zeta) - \phi_{cr}(\zeta)}{\phi_{cr}(\zeta)^{1/5}}. \quad (6.49)$$

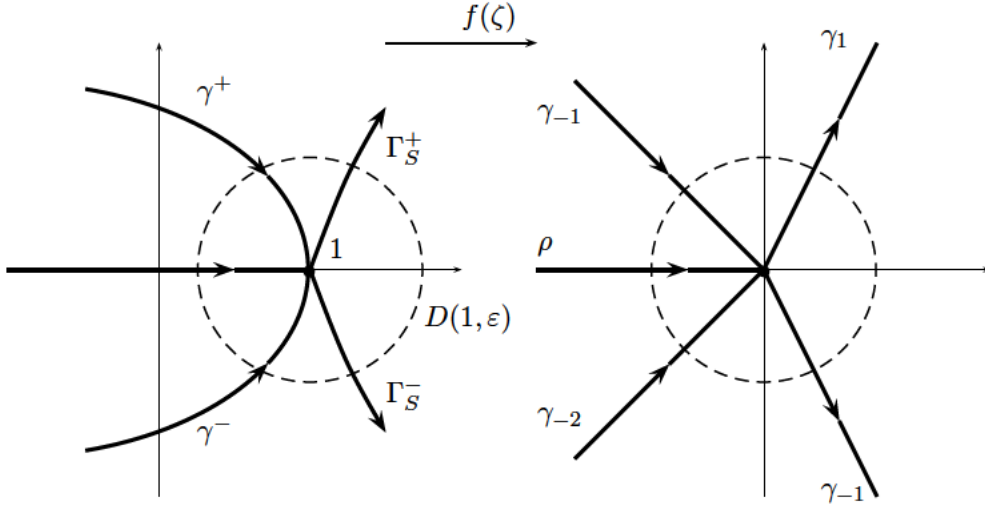
- $E : D(1, \varepsilon) \mapsto \mathbb{C}^{2 \times 2}$ is a suitable analytic prefactor:

$$E(\zeta) = M(\zeta) \left[\frac{(N^{2/5}f(\zeta))^{\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \right]^{-1} = \frac{1}{\sqrt{2}} M(\zeta) \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} (N^{2/5}f(\zeta))^{-\sigma_3/4}, \quad (6.50)$$

then $P(\zeta)$ is analytic in $D(1, \varepsilon) \setminus \Sigma_S$ and it satisfies the Riemann–Hilbert problem stated before.

Proof. The proof is based on the matrix $\tilde{P}(\zeta)$ presented before, in particular note the jumps (6.46). \square

We have the following result:


 FIGURE 7. Construction of the local parametrix in a neighborhood of $\zeta = 1$.

Lemma 6.2. *Suppose that $\lambda \in \mathbb{R}$ is not a pole of $y_\alpha(\lambda)$, and also*

$$N^{4/5}(u - u_c) = c_1 \lambda, \quad c_1 = 2^{-12/5} 3^{-7/4}. \quad (6.51)$$

Let $f(\zeta)$, $h_u(\zeta)$ and $E(\zeta)$ be as defined in (6.48), (6.49) and (6.50), then there is a disc around the point $\zeta = 1$, which depends only on λ , such that the previous properties and the matching condition (6.44) are satisfied.

Proof. We recall that

$$\phi_{cr}(\zeta) = 4 \int_1^\zeta (s+1)^{1/2} (s-1)^{3/2} ds, \quad (6.52)$$

and we can write

$$\phi_{cr}(\zeta) = \frac{8\sqrt{2}}{5} (\zeta - 1)^{5/2} (1 + v_{cr}(\zeta)), \quad (6.53)$$

where the function $v_{cr}(\zeta)$ is analytic in a neighborhood of $\zeta = 1$, with $v_{cr}(1) = 0$. Therefore, if we take a small enough disc around $\zeta = 1$, then $f(\zeta)$ given by (6.48) is a conformal map to a neighborhood of the origin. Actually,

$$f(\zeta) = 2^{1/5} (\zeta - 1) + \mathcal{O}((\zeta - 1)^2), \quad \zeta \rightarrow 1, \quad (6.54)$$

so in particular $f(\zeta)$ maps the interval $(1 - \varepsilon, 1)$ onto a part of the negative real axis.

The curve Γ_S^+ is that of steepest descent for $\phi_{cr}(\zeta)$, so on Γ_S^+ the function $\phi_{cr}(\zeta)$ is real and negative, i.e. $\arg \phi_{cr}(\zeta) = \pi$. Therefore, $\arg f(\zeta) = 2\pi/5$ when $\zeta \in \Gamma_S^+$.

Finally, the location of the lips of the lens inside the disc $D(1, \varepsilon)$ is chosen in such a way that they are mapped by $f(\zeta)$ onto part of the rays $\arg \zeta = \pm 4\pi/5$ (upper and lower lip respectively), taking into account (6.48). This is achieved by requiring that in a neighborhood of $\zeta = 1$, we have $\arg \phi_{cr}(\zeta) = \pm 2\pi$ on the upper and lower lips of the lens (respectively).

Regarding the properties of the map $h_u(\zeta)$, we can write

$$\phi_u(\zeta) - \phi_{cr}(\zeta) = (u - u_c) \phi^o(\zeta, u). \quad (6.55)$$

The function $\phi^o(\zeta, u)$ satisfies

$$\phi^o(\zeta, u) = 2^{7/2} 3^{7/4} (\zeta - 1)^{1/2} (1 + v^o(\zeta, u)), \quad (6.56)$$

where $v^o(\zeta, u)$ is analytic in ζ in a neighborhood of $\zeta = 1$ and $v^o(1, u) = 0$. This follows from formula (6.13), namely

$$\phi_u(\zeta) = 2\pi i \int_{\zeta}^1 r_u(s) ds, \quad (6.57)$$

where $r_u(s)$ is the analytic extension of $\varrho_u(s)$ to $\mathbb{C} \setminus (-\infty, 1]$, so

$$r_u(s) = \frac{m_u(s)\sqrt{s-\sigma_u}}{2\pi\sqrt{s-1}} = \frac{M_u(s)}{\sqrt{s-1}}, \quad (6.58)$$

with $M_u(s)$ analytic near $s = 1$. Also, note that from (6.53) we have

$$\phi_{cr}(\zeta)^{1/5} = 2^{7/10} 5^{-1/5} (\zeta - 1)^{1/2} (1 + \tilde{v}_{cr}(\zeta)), \quad (6.59)$$

with $\tilde{v}_{cr}(\zeta)$ analytic in a neighborhood of $\zeta = 1$ and $\tilde{v}_{cr}(1) = 0$. We conclude that

$$h^o(\zeta) := \left(\frac{1}{20}\right)^{1/5} \frac{\phi^o(\zeta)}{\phi_{cr}(\zeta)^{1/5}} = 2^{12/5} 3^{7/4} (1 + \mathcal{O}(\zeta - 1)), \quad (6.60)$$

is analytic in the variable ζ in a neighborhood of $\zeta = 1$. Observe that, differently from [18, Lemma 4.3], the error term in this case depends on u (actually on powers of $u - u_c$), but that does not change the properties of $h_u(\zeta)$ as a function of ζ locally near $\zeta = 1$.

As a consequence of the last formula, we can define $c_1^{-1} = h^o(1) = 2^{12/5} 3^{7/4}$, and then

$$N^{4/5} h_u(1) = N^{4/5} (u - u_c) h^o(1) = N^{4/5} (u - u_c) c_1^{-1} = \lambda. \quad (6.61)$$

We suppose that λ is not a pole of $y_\alpha(\lambda)$, so we can find a neighborhood $D(1, \varepsilon)$ of $\zeta = 1$, for ε small enough, such that $N^{4/5} h_u(\zeta)$ is not a pole of $y_\alpha(\lambda)$ for all $\zeta \in D(1, \varepsilon)$.

Finally, substituting formulas (6.40) and (6.41) for $M(\zeta)$, we have

$$E(\zeta) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \left(N^{2/5} f(\zeta) \frac{\zeta - \sigma_u}{\zeta - 1} \right)^{-\sigma_3/4}, \quad (6.62)$$

so it is an analytic function in a neighborhood of $\zeta = 1$, because $f(\zeta)$ has a simple zero at $\zeta = 1$, see (6.54). Moreover, using the asymptotic behavior of the $\Psi(\zeta; \lambda, \alpha)$ function, see (3.13), we have

$$P(\zeta) = E(\zeta) \frac{(N^{2/5} f(\zeta))^{\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \left(I + \mathcal{O}(N^{-1/5}) \right) e^{\theta(\zeta; \lambda) \sigma_3 - N \phi_u(\zeta) \sigma_3/2} \quad (6.63)$$

as $N \rightarrow \infty$. Substituting the expression for $\theta(\zeta; \lambda)$ and the formulas for $f(\zeta)$ and $h_u(\zeta)$, see (6.48) and (6.49), we obtain

$$\theta(\zeta; \lambda) = \theta(N^{2/5} f(\zeta); N^{4/5} h_u(\zeta)) = \frac{4}{5} (N^{2/5} f(\zeta))^{5/2} + N^{4/5} h_u(\zeta) (N^{2/5} f(\zeta))^{1/2} = \frac{N \phi_u(\zeta)}{2}, \quad (6.64)$$

so the exponential factor cancels out. Hence, substituting the expression for $E(\zeta)$, we get

$$P(\zeta) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \left(\frac{\zeta - 1}{\zeta + 1} \right)^{\sigma_3/4} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \left(I + \mathcal{O}(N^{-1/5}) \right) = M(\zeta) \left(I + \mathcal{O}(N^{-1/5}) \right), \quad (6.65)$$

uniformly for $\zeta \in \partial D(1, \varepsilon)$. □

6.6. Parametrix near the endpoint $\zeta = \sigma_u$. Near the endpoint $\zeta = \sigma_u$, we take a small disc $D(\sigma_u, \varepsilon) = \{\zeta : |\zeta - \sigma_u| < \varepsilon\}$, and the local parametrix should satisfy the following RH problem:

- (1) $Q(\zeta)$ is analytic in $\mathbb{C} \setminus \Gamma_S$, see Figure 6.
- (2) On Γ_S we have

$$Q_+(s) = Q_-(s) \begin{cases} \begin{pmatrix} 1 & e^{N\tilde{\phi}_u(s)} \\ 0 & 1 \end{pmatrix}, & s \in D(\sigma_u, \varepsilon) \cap (-\infty, \sigma_u], \\ \begin{pmatrix} 1 & 0 \\ e^{-N\phi_u(s)} & 1 \end{pmatrix}, & s \in D(\sigma_u, \varepsilon) \cap (\gamma^+ \cup \gamma^-), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & s \in D(\sigma_u, \varepsilon) \cap (\sigma_u, 1). \end{cases} \quad (6.66)$$

- (3) $Q(\zeta) = M(\zeta) (I + \mathcal{O}(N^{-1}))$, $N \rightarrow \infty$, uniformly for $\zeta \in \partial D(\sigma_u, \varepsilon)$.

Note that if we write

$$\tilde{Q}(\zeta) = Q(\zeta) e^{N\tilde{\phi}_u(\zeta)\sigma_3/2}, \quad (6.67)$$

then the matrix $\tilde{Q}(\zeta)$ satisfies the following Riemann–Hilbert problem:

- (1) $\tilde{Q}(\zeta)$ is analytic in $D(\sigma_u, \varepsilon) \setminus \Gamma_S$.
- (2) $\tilde{Q}(\zeta)$ has the following jumps:

$$J_{\tilde{Q}}(s; u) = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & s \in D(\sigma_u, \varepsilon) \cap (-\infty, \sigma_u], \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & s \in D(\sigma_u, \varepsilon) \cap (\gamma^+ \cup \gamma^-), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & s \in D(\sigma_u, \varepsilon) \cap (\sigma_u, 1). \end{cases} \quad (6.68)$$

- (3) Uniformly for $\zeta \in \partial D(\sigma_u, \varepsilon)$, we have

$$\tilde{Q}(\zeta) = M(\zeta) (I + \mathcal{O}(N^{-1})) e^{N\tilde{\phi}_u(\zeta)\sigma_3/2}, \quad N \rightarrow \infty, \quad (6.69)$$

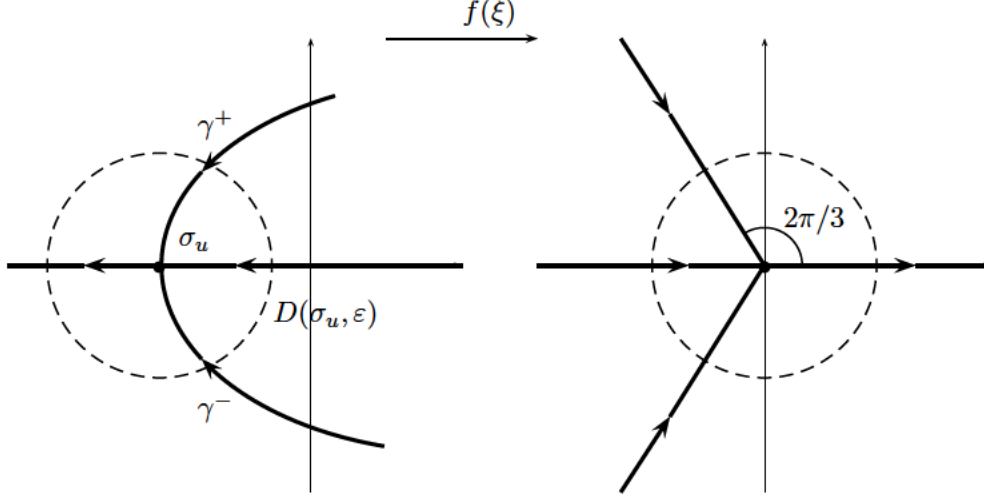
This problem can be solved in terms of Airy functions. We write $\xi = -\zeta$, and then

$$\tilde{Q}(\zeta) = \sigma_3 \tilde{Q}(\xi) \sigma_3, \quad (6.70)$$

where $\tilde{Q}(\xi)$ solves a standard Airy Riemann–Hilbert problem near the endpoint $-\sigma_u > 0$.

In order to construct the matrix the local parametrix, we consider the functions

$$y_0(\xi) = \text{Ai}(\xi), \quad y_1(\xi) = \omega \text{Ai}(\omega\xi), \quad y_2(\xi) = \omega^2 \text{Ai}(\omega^2\xi), \quad (6.71)$$


 FIGURE 8. Construction of the local parametrix in a neighborhood of $\zeta = \sigma_u$.

where $\omega = e^{2\pi i/3}$, which are solutions of the Airy differential equation $y''(\xi) = \xi y(\xi)$. We construct the following function:

$$A(\xi) = \begin{cases} \begin{pmatrix} y_0(\xi) & -y_2(\xi) \\ -iy'_0(\xi) & iy'_2(\xi) \end{pmatrix}, & \arg \xi \in (0, \frac{2\pi}{3}), \\ \begin{pmatrix} -y_1(\xi) & -y_2(\xi) \\ iy'_1(\xi) & iy'_2(\xi) \end{pmatrix}, & \arg \xi \in (\frac{2\pi}{3}, \pi), \\ \begin{pmatrix} -y_2(\xi) & y_1(\xi) \\ iy'_2(\xi) & -iy'_1(\xi) \end{pmatrix}, & \arg \xi \in (-\pi, -\frac{2\pi}{3}), \\ \begin{pmatrix} y_0(\xi) & y_1(\xi) \\ -iy'_0(\xi) & -iy'_1(\xi) \end{pmatrix}, & \arg \xi \in (-\frac{2\pi}{3}, 0). \end{cases} \quad (6.72)$$

This matrix-valued function satisfies the jumps in (6.68), on contours emanating from the origin with angles $\pm 2\pi/3$ and π , as a consequence of the relation $y_0(\xi) + y_1(\xi) + y_2(\xi) = 0$.

We also note the asymptotic behavior of the Airy function and its derivative, as the argument grows large:

$$A(w) = w^{-\sigma_3/4} \frac{1}{2\sqrt{\pi}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left(I + \mathcal{O}(w^{-3/2}) \right) e^{-\frac{2}{3}w^{3/2}\sigma_3}, \quad w \rightarrow \infty, \quad (6.73)$$

which follows from the standard asymptotic expansion of the Airy function and its derivative, see for instance [1, 9.7.5, 9.7.6].

Now we consider the following local parametrix:

$$\tilde{Q}(\xi) = E(\xi) A(N^{2/3} \tilde{f}(\xi)), \quad (6.74)$$

where $E(\xi)$ is an analytic prefactor, $A(w)$ solves the standard Airy Riemann–Hilbert problem and $w = N^{2/3} \tilde{f}(\xi)$ is a conformal map from $D(-\sigma_u, \varepsilon)$ to a neighborhood of the origin in the auxiliary w plane.

The function $w = N^{2/3} \tilde{f}(\xi)$ is determined in such a way that we obtain a matching between the exponential factors in (6.69) and (6.73):

$$-\frac{2}{3} \left[N^{2/3} \tilde{f}(\xi) \right]^{3/2} = \frac{1}{2} N \tilde{\phi}_u(\xi) \Rightarrow \tilde{f}(\xi) = \left(-\frac{3}{4} \tilde{\phi}_u(\xi) \right)^{2/3}. \quad (6.75)$$

This is a conformal mapping in a neighborhood of the point $-\sigma_u$ for ε and $u - u_c$ small enough, because of the properties of $\tilde{\phi}_u$.

Regarding the factor $E(\xi)$, we construct it in such a way that it takes care of the matching between the local and global parametrices:

$$E(\xi) = \sqrt{\pi} M(\xi) \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \left(N^{2/3} \tilde{f}(\xi) \right)^{\sigma_3/4}. \quad (6.76)$$

The fractional power $\tilde{f}(\xi)^{\sigma_3/4}$ has a jump on the interval $[-\sigma_u - \delta, -\sigma_u]$, that is compensated with the jump of $M(\xi)$, and it is not difficult to check that $E(\xi)$ is indeed analytic in a neighborhood of $\xi = -\sigma_u$. Thus, the local parametrix is

$$\begin{aligned} Q(\zeta) &= \tilde{Q}(\zeta) e^{-N \tilde{\phi}_u(\zeta) \sigma_3/2} \\ &= \sigma_3 \tilde{Q}(\zeta) \sigma_3 e^{-N \tilde{\phi}_u(\zeta) \sigma_3/2} \\ &= \sigma_3 E(\xi) A(N^{2/3} \tilde{f}(\xi)) \sigma_3 e^{-N \tilde{\phi}_u(\zeta) \sigma_3/2} \end{aligned} \quad (6.77)$$

We also note that the matrix $A(N^{2/3} \tilde{f}(\xi))$ admits a full asymptotic expansion in powers of N^{-1} , because of (6.73) and so does $\tilde{Q}(\zeta)$, uniformly for ζ in a neighborhood of $\zeta = \sigma_u$.

6.7. Third transformation of the RHP. The final step of the Riemann–Hilbert analysis is to define a matrix $R_u(\zeta)$ as follows:

$$R(\zeta) = S(\zeta) \begin{cases} M^{-1}(\zeta), & \zeta \in \mathbb{C} \setminus (\overline{D(1, \varepsilon)} \cup \overline{D(\sigma_u, \varepsilon)} \cup \Gamma_S), \\ P^{-1}(\zeta), & \zeta \in D(1, \varepsilon) \setminus \Gamma_S, \\ Q^{-1}(\zeta), & \zeta \in D(\sigma_u, \varepsilon) \setminus \Gamma_S, \end{cases} \quad (6.78)$$

where Γ_S is the contour shown in Figure 6. This matrix $R(\zeta)$ satisfies the following Riemann–Hilbert problem:

- (1) $R(\zeta)$ is analytic in $\mathbb{C} \setminus \Gamma_R$, where Γ_R is shown in Figure 9.
- (2) On Γ_R ,

$$R_+(s) = R_-(s) \begin{cases} M(s) \begin{pmatrix} 1 & e^{N \tilde{\phi}_u(s)} \\ 0 & 1 \end{pmatrix} M^{-1}(s), & s \in \Gamma_{R,1}, \\ M(s) \begin{pmatrix} 1 & 0 \\ e^{-N \phi_u(s)} & 1 \end{pmatrix} M^{-1}(s), & s \in \Gamma_{R,2} \cup \Gamma_{R,4}, \\ M(s) \begin{pmatrix} 1 & \alpha e^{N \phi_u(s)} \\ 0 & 1 \end{pmatrix} M^{-1}(s), & s \in \Gamma_{R,3}, \\ M(s) \begin{pmatrix} 1 & (1 - \alpha) e^{N \phi_u(s)} \\ 0 & 1 \end{pmatrix} M^{-1}(s), & s \in \Gamma_{R,5}, \\ P(s) M^{-1}(s), & s \in \partial D(1, \varepsilon), \\ Q(s) M^{-1}(s), & s \in \partial D(\sigma_u, \varepsilon), \end{cases} \quad (6.79)$$

(3) The matrix $R(\zeta)$ has the following asymptotic behavior:

$$R(\zeta) = I + \mathcal{O}(\zeta^{-1}), \quad \zeta \rightarrow \infty. \quad (6.80)$$

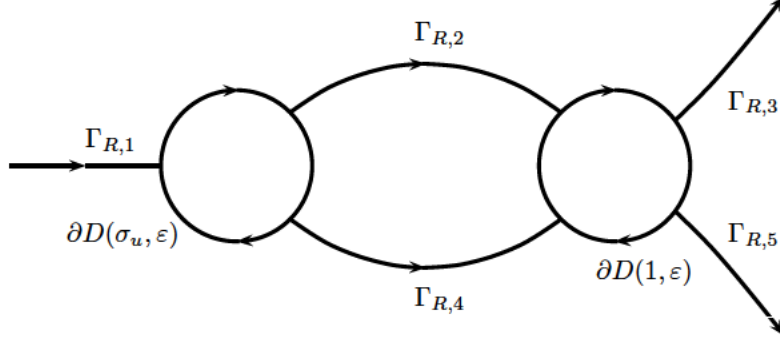


FIGURE 9. The contour of integration Γ_R .

From the matching conditions with $M(\zeta)$ and the last two equations in (6.79), we find that the jump matrix $J_R(s; u)$ can be expanded in inverse powers of $N^{-1/5}$ and N^{-1} on the boundary of the discs $D(1, \varepsilon)$ and $D(\sigma_u, \varepsilon)$ respectively. Namely, as $N \rightarrow \infty$,

$$J_R(\zeta; u) \sim \begin{cases} I + \sum_{k=1}^{\infty} W^{(k)}(\zeta; u) N^{-k/5}, & \zeta \in \partial D(1, \varepsilon) \\ I + \sum_{k=1}^{\infty} \widehat{W}^{(k)}(\zeta; u) N^{-k}, & \zeta \in \partial D(\sigma_u, \varepsilon) \end{cases} \quad (6.81)$$

The terms in these expansions can be written as follows:

$$W^{(k)}(\zeta; u) = M(\zeta) \Psi_k(N^{4/5} h_u(\zeta), \alpha) f(\zeta)^{-k/2} M^{-1}(\zeta), \quad k \geq 1 \quad (6.82)$$

and

$$\widehat{W}^{(k)}(\zeta; u) = M(\zeta) \sigma_3 A_k(N^{2/3} \tilde{f}(-\zeta)) \tilde{f}(-\zeta)^{-3k/2} \sigma_3 M^{-1}(\zeta), \quad k \geq 1. \quad (6.83)$$

As a consequence, the matrix $R(\zeta)$ itself admits an asymptotic expansion in powers of $N^{-1/5}$ that holds uniformly for $\zeta \in \mathbb{C} \setminus \Gamma_R$, where the contour Γ_R is depicted in Figure 9.

Proposition 6.3. *The Riemann–Hilbert problem for $R(\zeta)$ has a unique solution for large enough N , and the matrix $R(\zeta)$ can be expanded in asymptotic series*

$$R(\zeta) \sim I + \sum_{k=1}^{\infty} R^{(k)}(\zeta) \quad (6.84)$$

where $R^{(k)}(\zeta) = \mathcal{O}(N^{-k/5})$ as $N \rightarrow \infty$, uniformly for $\zeta \in \mathbb{C} \setminus \Gamma_R$.

Proof. The proof follows the lines of [6, §11], so we omit the details. Note that because of Lemma 6.1 and Lemma 6.2, the jump matrices on the contours $\Gamma_{R,j}$, $j = 1, \dots, 5$ are exponentially close to the identity. On the circles around the endpoints we have expansions in inverse powers of N and $N^{1/5}$, see (6.81). Thus the leading contribution is given by the expansion on $\partial D(1, \varepsilon)$. Alternatively, one can derive RH problems for higher order terms in the expansion of $R(\zeta)$, similarly to [18]. \square

7. PROOF OF THEOREM 4.1

As a consequence of the previous steepest descent analysis of the Riemann–Hilbert problem, we have proved the following:

- The Riemann–Hilbert problem is solvable for large enough values of N , and the solution $Y(\zeta)$ is unique. This follows from the solvability of the Riemann–Hilbert for $R(\zeta)$ and the fact that all the transformations $Y \mapsto T \mapsto S \mapsto R$ are invertible.
- In particular, the $(1, 1)$ entry of Y exists for large enough N , and this is the n -th orthogonal polynomial $P_n(\zeta)$ with respect to the weight function $e^{-NV(\zeta; u)}$, where $V(\zeta; u)$ is described in (5.15), and orthogonality is defined on the curve Γ described in (1.15).

Moreover, if $P_n(\zeta, u)$ and also $P_{n\pm 1}(\zeta, u)$ exist, then they satisfy a three term recurrence relation:

$$\zeta P_n(\zeta, u) = P_{n+1}(\zeta, u) + \beta_{n,N}(u)P_n(\zeta, u) + \gamma_{n,N}^2(u)P_{n-1}(\zeta, u). \quad (7.1)$$

In particular, this result is true in the diagonal case $n = N$, since we have a solution of the RH problem for large N . Then it is well known that the recurrence coefficients $\gamma_{N,N}^2(u)$ and $\beta_{N,N}(u)$ can be expressed in terms of the entries of the matrices in the Riemann–Hilbert analysis (for convenience, we omit the dependence on u in the matrices):

$$\begin{aligned} \gamma_{N,N}^2(u) &= [Y_1]_{12}[Y_1]_{21} = [T_1]_{12}[T_1]_{21}, \\ \beta_{N,N}(u) &= \frac{[Y_2]_{12}}{[Y_1]_{12}} - [Y_1]_{22} = \frac{[T_2]_{12}}{[T_1]_{12}} - [T_1]_{22}, \end{aligned} \quad (7.2)$$

where

$$Y(\zeta)\zeta^{-N\sigma_3} = I + \frac{Y_1}{\zeta} + \frac{Y_2}{\zeta^2} + \dots, \quad T(\zeta) = I + \frac{T_1}{\zeta} + \frac{T_2}{\zeta^2} + \dots, \quad \zeta \rightarrow \infty. \quad (7.3)$$

Now we recall that $T = S = RM$ outside the discs $D(1, \varepsilon)$ and $D(\sigma_u, \varepsilon)$ and the lens that we opened in the $T \mapsto S$ transformation, and we use the large ζ asymptotic expansion

$$\begin{aligned} M(\zeta) &= I + \frac{M_1}{\zeta} + \frac{M_2}{\zeta^2} + \dots \\ &= I + \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \frac{1 - \sigma_u}{4\zeta} + \begin{pmatrix} 1 - \sigma_u & 4i(1 + \sigma_u) \\ -4i(1 + \sigma_u) & 1 - \sigma_u \end{pmatrix} \frac{1 - \sigma_u}{32\zeta^2} + \dots, \quad \zeta \rightarrow \infty. \end{aligned} \quad (7.4)$$

Since

$$\begin{aligned} S_1 &= T_1 = M_1 + R_1, \\ S_2 &= T_2 = M_2 + R_1 M_1 + R_2, \end{aligned} \quad (7.5)$$

we obtain

$$\begin{aligned} \gamma_{N,N}^2(u) &= [M_1]_{12}[M_1]_{21} + [M_1]_{12}[R_1]_{21} + [M_1]_{21}[R_1]_{12} + [R_1]_{12}[R_1]_{21}, \\ &= [M_1]_{12}[M_1]_{21} + \mathcal{O}(N^{-1/5}), \\ &= \frac{(1 - \sigma_u)^2}{16} + \mathcal{O}(N^{-1/5}), \\ \beta_{N,N}(u) &= \frac{[M_2]_{12} + [R_1 M_1]_{12} + [R_2]_{12}}{[M_1]_{12} + [R_1]_{12}} - [M_1]_{22} + [R_1]_{22} \\ &= \frac{[M_2]_{12}}{[M_1]_{12}} - [M_1]_{22} + \mathcal{O}(N^{-1/5}) \\ &= \frac{1 + \sigma_u}{2} + \mathcal{O}(N^{-1/5}). \end{aligned} \quad (7.6)$$

More in general, because of the expansion of $R(\zeta)$ in inverse powers of $N^{1/5}$ obtained in Proposition 6.3, the coefficients $\gamma_{N,N}^2(u)$ and $\beta_{N,N}(u)$ can be expanded in inverse powers of $N^{1/5}$ as well. As a consequence of the Riemann–Hilbert analysis, we have the following asymptotic expansions:

$$\gamma_{N,N}^2(u) \sim \gamma_c^2 + \sum_{k=1}^{\infty} \frac{1}{N^{k/5}} p_k(\lambda), \quad \beta_{N,N}(u) \sim \beta_c + \sum_{k=1}^{\infty} \frac{1}{N^{k/5}} q_k(\lambda), \quad (7.7)$$

where γ_c^2 and β_c are the values of the coefficients at the critical time $u = u_c$, and u is related to λ by means of the double scaling relation $N^{4/5}(u - u_c) = c_1 \lambda$.

In order to prove that the odd terms in the previous expansions are 0, as claimed in the theorem, and also the connection with the solution to the Painlevé I differential equation, it is more convenient to use the double scaling in terms of the ratio n/N , see (4.8).

We can establish a simple analytic relation between the variables λ and v in a neighborhood of the origin, which corresponds to the critical case.

Lemma 7.1. *We can write*

$$\lambda(v) = N^{4/5} \varphi(v N^{-4/5}), \quad (7.8)$$

where $\varphi(t)$ is analytic in a neighborhood of $t = 0$ and $\varphi(0) = 0$. Furthermore, $\varphi'(0) \neq 0$, so

$$v(\lambda) = N^{4/5} \varphi^{-1}(\lambda N^{-4/5}), \quad (7.9)$$

the inverse function $\varphi^{-1}(t)$ being analytic in a neighborhood of $t = 0$.

Proof. We can express the variable λ in terms of v :

$$1 + v N^{-4/5} = \frac{n}{N} = \frac{u^2}{u_c^2} = (1 + c_2 \lambda N^{-4/5})^2 = \left(1 + c_2 \lambda \left(\frac{n}{N}\right)^{-4/5} N^{-4/5}\right)^2, \quad (7.10)$$

where we recall that $c_2 = 2^{-7/5}$, and hence for bounded v and large enough N we can take square root and write:

$$\lambda(v) = N^{4/5} \varphi(N^{-4/5} v), \quad (7.11)$$

where the function

$$\varphi(t) = \frac{\sqrt{1+t} - 1}{c_2} (1+t)^{4/5} \quad (7.12)$$

is analytic in a neighborhood of $t = 0$. The first few terms in the Taylor expansion give

$$\lambda(v) = \frac{1}{2c_2} v + \frac{11}{40c_2} v^2 N^{-4/5} + \mathcal{O}(N^{-8/5}), \quad (7.13)$$

so $\varphi'(0) = 2^{2/5} \neq 0$, and by the inverse function theorem, locally near $\lambda = 0$ we can write

$$v(\lambda) = N^{4/5} \varphi^{-1}(N^{-4/5} \lambda), \quad (7.14)$$

where $\varphi^{-1}(t)$ is analytic in a neighborhood of $t = 0$. By perturbation of (7.13), the first few terms of this expansion are

$$v(\lambda) = 2c_2 \lambda - \frac{11}{5} c_2^2 \lambda^2 N^{-4/5} + \mathcal{O}(N^{-8/5}). \quad (7.15)$$

□

Corollary 7.2. *The quantity n/N can be expanded as a power series in $N^{-4/5}$, which is convergent for large enough N .*

Proof. This result follows directly from the previous lemma. Observe that the series expansion for $v(x)$ in (7.14) is also a series in powers of $N^{-4/5}$, provided that λ is bounded. □

Now consider the recurrence coefficients corresponding to the new weight $e^{-NV(\xi; u_c)}$, where $V(\xi; u_c) = \frac{\xi^2}{2} - u_c \xi^3$. We denote these coefficients by $\gamma_{N,n}^2(u_c)$ and $\beta_{N,n}(u_c)$, and we have

$$\gamma_{N,n}^2(u_c) = \frac{u^2}{u_c^2} \gamma_{N,N}^2(u), \quad \beta_{N,n}(u_c) = \frac{u}{u_c} \beta_{N,N}(u). \quad (7.16)$$

As a consequence of the Riemann–Hilbert analysis, we have an asymptotic expansion in inverse powers of $N^{1/5}$ of the coefficients $\gamma_{N,N}^2(u)$ and $\beta_{N,N}(u)$, see (7.7). Because of formulas (7.16) and the connection between u/u_c and n/N given before, we obtain

$$\begin{aligned} \gamma_{N,n}^2(u_c) &\sim \left[\gamma_c^2 + \sum_{k=1}^{\infty} \frac{1}{N^{k/5}} p_k(x) \right] \left(1 + \frac{c_2 \lambda}{N^{4/5}} \right)^2, \\ \beta_{N,n}(u_c) &\sim \left[\beta_c + \sum_{k=1}^{\infty} \frac{1}{N^{k/5}} q_k(x) \right] \left(1 + \frac{c_2 \lambda}{N^{4/5}} \right). \end{aligned} \quad (7.17)$$

Using the double scaling and the relation between x and v from the previous lemma, we can write an expansion in inverse powers of $N^{1/5}$:

$$\begin{aligned} \gamma_{N,n}^2(u_c) &\sim \left[\gamma_c^2 + \sum_{k=1}^{\infty} \frac{p_k(N^{4/5} \varphi(v N^{-4/5}))}{N^{k/5} (1 + v N^{-4/5})^k} \right] \left(1 + \frac{c_2 \varphi(v N^{-4/5})}{(1 + v N^{-4/5})^{4/5}} \right)^2 = \gamma_c^2 + \sum_{k=1}^{\infty} \frac{1}{N^{k/5}} \hat{p}_k(v), \\ \beta_{N,n}(u_c) &\sim \left[\beta_c + \sum_{k=1}^{\infty} \frac{q_k(N^{4/5} \varphi(v N^{-4/5}))}{N^{k/5} (1 + v N^{-4/5})^k} \right] \left(1 + \frac{c_2 \varphi(v N^{-4/5})}{(1 + v N^{-4/5})^{4/5}} \right) = \beta_c + \sum_{k=1}^{\infty} \frac{1}{N^{k/5}} \hat{q}_k(v), \end{aligned} \quad (7.18)$$

for some coefficients $\hat{p}_k(v)$ and $\hat{q}_k(v)$ that can be computed from the previous formula, in terms of p_k and q_k , using the analyticity of the coefficients p_k and q_k and of the function φ . Explicitly, the first coefficients are

$$\begin{aligned} \hat{p}_k(v) &= p_k \left(\frac{v}{2c_2} \right) = p_k(2^{-2/5} v), \quad k = 1, 2, 3, 4, \\ \hat{q}_k(v) &= q_k \left(\frac{v}{2c_2} \right) = q_k(2^{-2/5} v), \quad k = 1, 2, 3, 4. \end{aligned} \quad (7.19)$$

Higher order coefficients can be computed, using p_k , q_k and their derivatives. Next we consider the asymptotic expansions (7.18) together with the string equations corresponding to the weight $e^{-NV(\xi; u_c)}$ (we omit the dependence on u_c for brevity):

$$\begin{aligned} 3u_c(\hat{\gamma}_{N,n}^2 + \hat{\beta}_{N,n}^2 + \hat{\gamma}_{N,n+1}^2) &= \hat{\beta}_{N,n}, \\ \hat{\gamma}_{N,n}^2(1 - 3u_c(\hat{\beta}_{N,n} + \hat{\beta}_{N,n-1})) &= \frac{n}{N}. \end{aligned} \quad (7.20)$$

We note the symmetry of the first string equation with respect to the change of indices

$$\sigma_0 = \{\hat{\gamma}_{N,j}^2 \rightarrow \hat{\gamma}_{N,2n+1-j}^2, \hat{\beta}_{N,j} \rightarrow \hat{\beta}_{N,2n-j}, j = 0, 1, \dots\}, \quad (7.21)$$

and the second string equation with respect to

$$\sigma_1 = \{\hat{\gamma}_{N,j}^2 \rightarrow \hat{\gamma}_{N,2n-j}^2, \hat{\beta}_{N,j} \rightarrow \hat{\beta}_{N,2n-1-j}, j = 0, 1, 2, \dots\}. \quad (7.22)$$

As a consequence, all odd terms in the expansion are 0, and in fact we can write

$$\hat{\gamma}_{N,n}^2(u_c) \sim \gamma_c^2 + \sum_{k=1}^{\infty} \frac{1}{N^{2k/5}} \hat{p}_{2k}(v), \quad \hat{\beta}_{N,n}(u_c) \sim \beta_c + \sum_{k=1}^{\infty} \frac{1}{N^{2k/5}} \hat{q}_{2k}(v). \quad (7.23)$$

The complete proof of this property can be found in [8, Section 5].

The asymptotic expansions for the coefficients $\gamma_{N,N}^2(u)$ and $\beta_{N,N}(u)$ in inverse powers of $N^{2/5}$ now follow directly from (7.23) and (7.16):

$$\gamma_{N,N}(u) \sim \left[\gamma_c^2 + \sum_{k=1}^{\infty} \frac{\hat{p}_{2k}(v)}{N^{2k/5}} \right] \frac{1}{(1 + c_2 \lambda N^{-4/5})^2} = \gamma_c^2 + \sum_{k=1}^{\infty} \frac{p_{2k}(\lambda)}{N^{2k/5}}, \quad (7.24)$$

for some coefficients $p_{2k}(\lambda)$. Here we have used the change of variables from v to λ and also

$$\frac{n}{N} = 1 + v N^{-4/5} = 1 + \varphi^{-1}(\lambda N^{-4/5}). \quad (7.25)$$

The first coefficients are

$$p_2(\lambda) = \hat{p}_2(2c_2\lambda) = \hat{p}_2(2^{-2/5}\lambda), \quad p_4(\lambda) = \hat{p}_4(2c_2\lambda) = \hat{p}_4(2^{-2/5}\lambda). \quad (7.26)$$

Similarly,

$$\beta_{N,N}(u) \sim \left[\beta_c + \sum_{k=1}^{\infty} \frac{\hat{q}_{2k}(v)}{N^{2k/5}} \right] \frac{1}{1 + c_2 \lambda N^{-4/5}} \sim \beta_c + \sum_{k=1}^{\infty} \frac{q_{2k}(\lambda)}{N^{2k/5}}, \quad (7.27)$$

for some coefficients $q_{2k}(\lambda)$. The first terms are

$$q_2(\lambda) = \hat{q}_2(2c_2\lambda) = \hat{q}_2(2^{-2/5}\lambda), \quad q_4(\lambda) = \hat{q}_4(2c_2\lambda) = \hat{q}_4(2^{-2/5}\lambda). \quad (7.28)$$

In order to identify the coefficients $p_2(\lambda)$ and $q_2(\lambda)$ in terms of Painlevé I functions, we take the first few terms from the asymptotic expansion (7.23):

$$\begin{aligned} \gamma_{N,n}^2(u_c) &= \gamma_c^2 + N^{-2/5} \hat{p}_2(v) + N^{-4/5} \hat{p}_4(v) + \dots, \\ \beta_{N,n}(u_c) &= \beta_c + N^{-2/5} \hat{q}_2(\tilde{v}) + N^{-4/5} \hat{q}_4(\tilde{v}) + \dots, \end{aligned} \quad (7.29)$$

Now we substitute this into the string equations (7.20) and consider different powers of N . At the zeroth level we have the following relations for $\hat{p}_0 = \gamma_c^2$ and $\hat{q}_0 = \beta_c$:

$$72u_c^2 \hat{p}_0^3 - \hat{p}_0^2 + 1 = 0, \quad \hat{q}_0 = \frac{\hat{p}_0 - 1}{6u_c \hat{p}_0}. \quad (7.30)$$

The cubic equation for \hat{p}_0 has a double root

$$\gamma_c^2 = \sqrt{3} \quad (7.31)$$

and one simple root $\gamma_{c*}^2 = -\frac{\sqrt{3}}{2}$. The critical value for \hat{q}_0 can be computed from the second equation in (7.30):

$$\beta_c = \frac{\gamma_c^2 - 1}{6u_c \gamma_c^2} = 3^{1/4}(\sqrt{3} - 1). \quad (7.32)$$

These critical values can also be computed from the value of the coefficients $g_0(u_c)$ and $b_0(u_c)$ coming from the regular regime, see [7, §7].

Next we have factors multiplying $N^{-2/5}$. We consider the first string equation and expand around \tilde{v} . Observe that

$$\begin{aligned} \hat{\gamma}_{N,n}^2(u_c) &= \gamma_c^2 + N^{-2/5} \hat{p}_2 \left(\tilde{v} - \frac{N^{-1/5}}{2} \right) + N^{-4/5} \hat{p}_4 \left(\tilde{v} - \frac{N^{-1/5}}{2} \right) + \dots, \\ \hat{\gamma}_{N,n+1}^2(u_c) &= \gamma_c^2 + N^{-2/5} \hat{p}_2 \left(\tilde{v} + \frac{N^{-1/5}}{2} \right) + N^{-4/5} \hat{p}_4 \left(\tilde{v} + \frac{N^{-1/5}}{2} \right) + \dots, \end{aligned} \quad (7.33)$$

so when we expand in Taylor series around \tilde{v} and add these two quantities, only the even terms remain:

$$\hat{\gamma}_n^2 + \hat{\gamma}_{n+1}^2 = 2\gamma_c^2 + 2N^{-2/5} \hat{p}_2(\tilde{v}) + N^{-4/5} \left(\frac{1}{4} \hat{p}_2''(\tilde{v}) + 2\hat{p}_4(\tilde{v}) \right) + \dots, \quad (7.34)$$

and then we have from the first string equation

$$6u_c\hat{p}_2(\tilde{v}) - (1 - 6u_c\beta_c)\hat{q}_2(\tilde{v}) = 0. \quad (7.35)$$

Since $1 - 6u_c\beta_c = 1/\sqrt{3}$, see (7.32), we obtain

$$\hat{q}_2(\tilde{v}) = 3^{-1/4}\hat{p}_2(\tilde{v}). \quad (7.36)$$

Now we want to balance the terms multiplying $N^{-4/5}$, and that gives

$$\frac{1}{8}\hat{p}_2''(\tilde{v}) + \frac{1}{2}\hat{q}_2^2(\tilde{v}) = -\hat{p}_4(\tilde{v}) + 3^{1/4}\hat{q}_4(\tilde{v}). \quad (7.37)$$

Consider now the second string equation in (7.20), we expand around v . From (7.29) we have

$$\begin{aligned} \hat{\beta}_{N,n}(u_c) &= \beta_c + N^{-2/5}\hat{q}_2\left(v + \frac{N^{-1/5}}{2}\right) + N^{-4/5}\hat{q}_4\left(v + \frac{N^{-1/5}}{2}\right) + \dots \\ \hat{\beta}_{N,n-1}(u_c) &= \beta_c + N^{-2/5}\hat{q}_2\left(v - \frac{N^{-1/5}}{2}\right) + N^{-4/5}\hat{q}_4\left(v - \frac{N^{-1/5}}{2}\right) + \dots, \end{aligned} \quad (7.38)$$

hence

$$\hat{\beta}_n + \hat{\beta}_{n-1} = 2\beta_c + 2N^{-2/5}\hat{q}_2(v) + N^{-4/5}\left(\frac{1}{4}\hat{q}_2''(v) + 2\hat{q}_4(v)\right) + \dots \quad (7.39)$$

By substituting this expression into the second equation in (7.20), we obtain the following equation for the terms multiplying $n^{-2/5}$:

$$(1 - 6u_c\beta_c)\hat{p}_2(v) - 6u_c\gamma_c^2\hat{q}_2(v) = 0, \quad (7.40)$$

which gives $\hat{q}_2(v) = 3^{-1/4}\hat{p}_2(v)$ again. If we group terms that multiply $N^{-4/5}$, we get

$$-\frac{3^{1/4}}{8}\hat{q}_2''(v) - 3^{-1/4}\hat{p}_2(v)\hat{q}_2(v) - 3^{1/2}v = -\hat{p}_4(v) + 3^{1/4}\hat{q}_4(v). \quad (7.41)$$

Let us compare equations (7.37) and (7.41). These are two equations on the functions $\hat{p}_2(v)$, $\hat{q}_2(v)$, $\hat{p}_4(v)$, and $\hat{q}_4(v)$. The fact that in (7.37) we use the variable \tilde{v} , and not v , does not matter, because these are equations on functions, and we can replace \tilde{v} by v in (7.37). Since the right hand sides of equations (7.37), (7.41) are equal, the left hand sides are equal as well:

$$-\frac{3^{1/4}}{8}\hat{q}_2''(v) - 3^{-1/4}\hat{p}_2(v)\hat{q}_2(v) - 3^{1/2}v = \frac{1}{8}\hat{p}_2''(\tilde{v}) + \frac{1}{2}\hat{q}_2^2(\tilde{v}). \quad (7.42)$$

We can reduce this to an equation on $\hat{p}_2(v)$ only:

$$\hat{p}_2''(v) = -2\sqrt{3}\hat{p}_2^2(v) - 4\sqrt{3}v, \quad (7.43)$$

which is the Painlevé I equation. Applying the change of variables

$$v = 2^{-2/5}\lambda, \quad \hat{p}_2(v) = -2^{4/5}3^{1/2}y_\alpha(\lambda), \quad (7.44)$$

we bring it to the standard form:

$$y_\alpha''(\lambda) = 6y_\alpha^2(\lambda) + \lambda. \quad (7.45)$$

We know that $y_\alpha(\lambda)$ is the appropriate solution of Painlevé I because of the asymptotic behavior coming from the Riemann–Hilbert analysis. Finally, from (7.26), we have

$$\hat{p}_2(2^{-2/5}\lambda) = p_2(\lambda) = -2^{4/5}3^{1/2}y(\lambda), \quad (7.46)$$

and also $q_2(\lambda) = \hat{q}_2(2^{-2/5}\lambda) = 3^{-1/4}\hat{p}_2(2^{-2/5}\lambda) = 3^{-1/4}p_2(\lambda)$, which concludes the proof.

8. PROOF OF THEOREM 4.4

From [7], we know that in the regular case (when $u_c - u \geq \delta$, for some fixed δ independent of N), the coefficient $\gamma_{N,N}^2(u)$ admits an asymptotic expansion in inverse powers of N^2 . On the other hand, Theorem 4.1 gives an asymptotic expansion in inverse powers of $N^{2/5}$ in the double scaling regime. The issue now is how to cover the gap between the regular and the double scaling cases in the asymptotics of $\gamma_{N,N}^2(u)$, by extending the regions of validity of those asymptotic expansions.

8.1. Extension of the regular regime. In the regular case, in terms of the parameter $w = u^2$ and setting $n = N$, we have the following asymptotic expansions, see [7]:

$$\hat{\gamma}_{N,N}^2(w) \sim \sum_{k=0}^{\infty} \frac{u^{4k}}{N^{2k}} \hat{g}_{2k}(w), \quad \hat{\beta}_{N,N}(w) \sim \sum_{k=0}^{\infty} \frac{u^{4k}}{N^{2k}} \hat{b}_{2k}(w), \quad (8.1)$$

where

$$\hat{g}_{2k}(w) = u^{-4k+2} g_{2k}(u), \quad \hat{b}_{2k}(w) = u^{-4k+1} b_{2k}(u), \quad k \geq 0. \quad (8.2)$$

These asymptotic expansions are uniform in u if $0 < u \leq u_c - \delta$, for fixed $\delta > 0$. This is a consequence of the Riemann–Hilbert analysis. The purpose of this section is to extend this property to the case when we let $u - u_c$ decrease with N , at a certain (slow enough) rate. To this end, we will need a local analysis near $\zeta = 1$ in a neighborhood that shrinks with N .

In the regular case, we recall the global parametrix:

$$M(\zeta) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \beta^{-1}(\zeta) & 0 \\ 0 & \beta(\zeta) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad \beta(\zeta) = \left(\frac{\zeta+1}{\zeta-1} \right)^{1/4}. \quad (8.3)$$

We construct the local parametrix $P(\zeta)$ in a neighborhood of $\zeta = 1$ in terms of Airy functions. Namely, given a disc $D(1, \varepsilon)$ and Γ_S the union of contours shown in Figure 10, the function $P(\zeta)$ satisfies the following RH problem:

- (1) $P(\zeta)$ is analytic in $D(1, \varepsilon) \setminus \Gamma_S$, and for every $x \in \Gamma_S$ the following limits exist:

$$P_{\pm}(s) = \lim_{\zeta \rightarrow s, \zeta \in \Omega_{\pm}} P(\zeta), \quad (8.4)$$

- (2) On Γ_S , we have the following jumps:

$$P_+(s) = P_-(s) \begin{cases} \begin{pmatrix} 1 & e^{N\phi_u(s)} \\ 0 & 1 \end{pmatrix}, & s \in D(1, \varepsilon) \cap [1, \infty) \\ \begin{pmatrix} 1 & 0 \\ e^{-N\phi_u(s)} & 1 \end{pmatrix}, & s \in D(1, \varepsilon) \cap (\gamma_+ \cup \gamma_-) \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & s \in D(1, \varepsilon) \cap [\sigma_u, 1] \end{cases} \quad (8.5)$$

- (3) Uniformly for $\zeta \in \partial D(1, \varepsilon)$,

$$P(\zeta) = M(\zeta) \left(I + \mathcal{O}\left(\frac{1}{N}\right) \right). \quad (8.6)$$

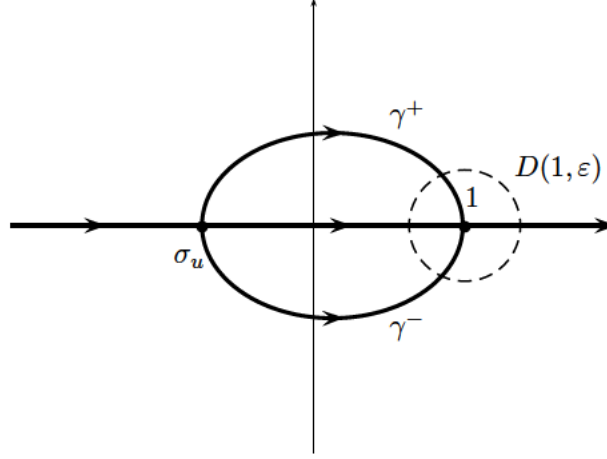
Define

$$\tilde{P}(\zeta) = P(\zeta) e^{N\phi_u(\zeta)\sigma_3/2}, \quad (8.7)$$

then we have

- (1) $\tilde{P}(\zeta)$ is analytic in $D(1, \varepsilon) \setminus \Gamma_S$, and for every $x \in \Gamma_S$ the following limits exist:

$$\tilde{P}_{\pm}(s) = \lim_{\zeta \rightarrow s, \zeta \in \Omega_{\pm}} \tilde{P}(\zeta), \quad (8.8)$$


 FIGURE 10. Contour Γ_S for the extension of the regular regime.

(2) On Γ_S , we have the following jumps:

$$\tilde{P}_+(s) = \tilde{P}_-(s) \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & s \in D(1, \varepsilon) \cap [1, \infty) \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & s \in D(1, \varepsilon) \cap (\gamma_+ \cup \gamma_-) \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & s \in D(1, \varepsilon) \cap [\sigma_u, 1] \end{cases} \quad (8.9)$$

(3) Uniformly for $\zeta \in \partial D(1, \varepsilon)$,

$$\tilde{P}(\zeta) = M(\zeta) \left(I + \mathcal{O} \left(\frac{1}{N^{\varepsilon_3}} \right) \right) e^{N\phi_u(\zeta)\sigma_3/2}, \quad (8.10)$$

for $\varepsilon_3 > 0$.

Consequently, we look for a local parametrix in the form

$$\tilde{P}(\zeta) = E(\zeta) A(N^{2/3} f(\zeta)), \quad (8.11)$$

with the analytic prefactor

$$E(\zeta) = M(\zeta) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^{-1} (N^{2/3} f(\zeta))^{\sigma_3/4}. \quad (8.12)$$

The matrix $A(N^{2/3} f(\zeta))$ solves the standard Airy–Riemann–Hilbert problem in an auxiliary $w = N^{2/3} f(\zeta)$ plane, and the function $f(\zeta)$ is

$$f(\zeta) = \left(-\frac{3}{4} \phi_u(\zeta) \right)^{2/3} = \left(-\frac{3\pi i}{2} \int_{\zeta}^1 \varrho_u(s) ds \right)^{2/3}, \quad \zeta \in \mathbb{C} \setminus (-\infty, 1], \quad (8.13)$$

where $\varrho_u(s)$ is the density of the equilibrium measure given by (5.1).

If we expand in powers of $s - 1$ and integrate, we obtain

$$\phi_u(\zeta) = C_u(\zeta - 1)^{3/2} \left(1 + \frac{3\zeta_0 - 15}{20(\zeta_0 - 1)}(\zeta - 1) + \mathcal{O}((\zeta - 1)^2) \right), \quad (8.14)$$

where

$$C_u = \frac{3\sqrt{2}}{16}u(b-a)^3(\zeta_0 - 1) > 0. \quad (8.15)$$

Recall that $\zeta_0 > 1$ if $0 \leq u < u_c$, as a consequence of [7, Prop. 2.2]. If $\zeta_0 = 1 + \delta$, with $\delta > 0$, then the root of the linear factor in the previous expansion is

$$\zeta^* = 1 - \frac{20(\zeta_0 - 1)}{3\zeta_0 - 15} = 1 + \frac{5}{3}\delta + \mathcal{O}(\delta^2) \quad (8.16)$$

This proves that $f(\zeta)$ is a conformal mapping of a neighborhood of $\zeta = 1$ onto a neighborhood of the origin in the w plane.

If we define now $R(\zeta) = S(\zeta)M(\zeta)^{-1}$ outside the discs $D(\pm 1, \varepsilon)$, and $R(\zeta) = S(\zeta)\tilde{P}(\zeta)^{-1}$ inside, then the jump for this matrix $R(\zeta)$ on the boundary of the discs is $J_R(\zeta) = M(\zeta)\tilde{P}(\zeta)^{-1}$, and because of the matching between $\tilde{P}(\zeta)$ and $M(\zeta)$, this jump is equal to $I + \mathcal{O}(1/N)$. This is used to prove that $R(\zeta)$ itself can be expanded in inverse powers of N , beginning with I , and this in turn is used to prove that the recurrence coefficients $\gamma_{N,N}^2(u)$ and $\beta_{N,N}(u)$ admit an asymptotic expansion in powers of $1/N$ too. This expansion is uniform in u for $0 < u \leq u_c - \delta$.

It is clear that if $\zeta_0 = 1 + \delta$, then we can find a (fixed) neighborhood of $\zeta = 1$ where the linear factor in (8.14) is positive. The problem that we will encounter when trying to extend the regular case closer to the critical value $u = u_c$ is that the single root of the density $\varrho_u(\zeta)$ can get arbitrarily close to $\zeta = 1$, and that spoils the analyticity of $f(\zeta)$ in a neighborhood of $\zeta = 1$. More precisely, if $\zeta_0 = 1 + cN^{-\gamma}$, for some $\gamma > 0$ and $c > 0$, then the linear factor before will vanish at the point

$$\zeta^* = 1 + \frac{5c}{4}N^{-\gamma} + \mathcal{O}(N^{-2\gamma}), \quad (8.17)$$

and $f(\zeta)$ will not be analytic in any fixed neighborhood of $\zeta = 1$ for N large enough. To get analyticity in this context, we construct a shrinking neighborhood of $\zeta = 1$.

We make the change of variables $s = 1 + \xi N^{-\gamma}$ and $\zeta = 1 + \tau N^{-\gamma}$, then we obtain

$$\phi_u(\tau) = C_u N^{-5\gamma/2} \tau^{3/2} \left(\tau - \frac{20c}{12 - 3cN^{-\gamma}} + \mathcal{O}(\tau^2) \right). \quad (8.18)$$

Now the linear factor is harmless when $c > 0$, since for any $N \geq 1$ and $\gamma > 0$ we can bound

$$\frac{20c}{12 - 3cN^{-\gamma}} > \frac{5c}{3}, \quad (8.19)$$

and then in any disc around $\tau = 0$ of radius $\delta < \frac{5c}{3}$, the function

$$N^{2/3}f(\tau) = N^{2/3} \left(-\frac{3}{4}\phi_u(\tau) \right)^{2/3} = C_u^{3/2} N^{\frac{2-5\gamma}{3}} \tau \left(\tau - \frac{20c}{12 - 3cN^{-\gamma}} + \mathcal{O}(\tau^2) \right)^{2/3} \quad (8.20)$$

will be analytic.

Because of the shrinking neighborhood, we get an extra factor $N^{\frac{2-5\gamma}{3}}$. In order to be able to do the matching with the global parametrix, we need the argument of the Airy function to grow large with N , and that leads to the condition

$$\frac{2 - 5\gamma}{3} > 0 \Rightarrow \gamma < \frac{2}{5}. \quad (8.21)$$

Hence, we suppose that $\gamma = \frac{2}{5} - \frac{\varepsilon_2}{2}$, with $\varepsilon_2 > 0$, so $\zeta_0 = 1 + cN^{-\frac{2}{5} + \frac{\varepsilon_2}{2}}$. Bearing in mind (5.14) and the fact that the variables u and s are essentially equivalent near the critical value, we conclude that we can cover the following range with the regular case:

$$u - u_c \sim \tilde{c}_3 N^{-\frac{4}{5} + \varepsilon_2}. \quad (8.22)$$

We note that in this setting we still obtain an asymptotic expansion for $R(\zeta)$ that is uniform in u , provided that (8.22) is satisfied. The difference is that the coefficients in the asymptotic expansion coming from the Airy parametrix will now depend on N , and actually grow as N gets large:

$$A(N^{2/3}f(\zeta)) = (N^{2/3}f(\zeta))^{-\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left(I + \sum_{k=1}^{\infty} \frac{a_k N^{k(5\gamma/2-1)}}{\tau^{3k/2}} \right) e^{-N\phi_u(\zeta)\sigma_3/2}. \quad (8.23)$$

However, because of the restriction (8.21), the exponent of N is still < 0 for every $k \geq 1$, so we obtain an asymptotic series, uniform in u in this extended regular regime. Note that the term in parenthesis is

$$I + \mathcal{O}(N^{5\gamma/2-1}) = I + \mathcal{O}(N^{-5\varepsilon_2/4}) \quad (8.24)$$

if $\gamma = \frac{2}{5} - \frac{\varepsilon_2}{2}$, with $\varepsilon_2 > 0$, and this has an effect on the matching condition (8.10).

8.2. Extension of the double scaling regime. In the double scaling regime, the asymptotic expansion for $\gamma_{N,N}(u)$ was obtained applying the Deift–Zhou nonlinear steepest descent to the Riemann–Hilbert problem. In particular, in the final step, the form of the asymptotic expansion for $R(\zeta)$ comes from the local parametrix near $\zeta = 1$, given in terms of the solution to the Painlevé I equation.

Observe that the uniform asymptotic expansion for $\Phi(\zeta; \lambda, \alpha)$ in (3.38) is obtained by considering the function $g(\zeta)$ instead of $\theta_0(\zeta)$. Bearing in mind (3.36), the first term that we neglect is

$$a_0(\zeta, \lambda) = -\frac{\sqrt{6}}{9}(-\lambda)^{5/4}\zeta^{-1/2}, \quad (8.25)$$

and we need this to be small as we let $\lambda \rightarrow \infty$. In the construction of the local parametrix, see (6.47), observe that $\zeta = \mathcal{O}(N^{2/5})$, and we suppose now that $-\lambda = \mathcal{O}(N^\varepsilon)$, for some $\varepsilon > 0$. In order for $a_0(\zeta, \lambda)$ to go to 0 as $N \rightarrow \infty$, we need

$$\frac{5\varepsilon}{4} - \frac{1}{5} < 0 \Rightarrow \varepsilon < \frac{4}{25}, \quad (8.26)$$

so the extension of the double scaling regime is possible provided that $(-\lambda) = \mathcal{O}(N^{4/25-\varepsilon})$ as $N \rightarrow \infty$. As a consequence, one can allow $u - u_c = \mathcal{O}(N^{-16/25-\varepsilon})$, as $N \rightarrow \infty$, so there is indeed an overlap between both regimes.

9. PROOF OF THEOREM 4.6

In order to analyze the free energy near the critical case, we will use the Toda equation, similarly to [7]:

$$\frac{d^2 \tilde{F}_N(t)}{dt^2} = \tilde{\gamma}_{N,N}^2(t), \quad (9.1)$$

expressed in terms of the parameter t , which is related to u as follows:

$$t = \frac{1}{4(3u)^{4/3}}. \quad (9.2)$$

Accordingly, the new critical value is $t_c = 3 \cdot 2^{-2/3}$. We recall from [7, Proposition 5.1] that the Toda equation (9.1) holds for any $t > t_c$.

The corresponding weight function is $\tilde{V}(w; t) = -\frac{w^3}{3} + tw$. More precisely, if we apply the change of variables

$$z = (3u)^{-1/3}w + \frac{1}{6u}, \quad (9.3)$$

where we assume that $u > 0$ and $(3u)^{-1/3} > 0$, we have, by straightforward algebra, that

$$V(z; u) - \frac{1}{108u^2} = \tilde{V}(w; t). \quad (9.4)$$

We observe that the recurrence coefficients in the t parameter, that we denote $\tilde{\gamma}_{N,N}(t)$ and $\tilde{\beta}_{N,N}(t)$, can be obtained from the original ones in the u parameter as follows:

$$\begin{aligned} \tilde{\gamma}_{N,N}^2(t) &= (3u)^{2/3} \gamma_{N,N}^2(u) = \frac{\gamma_{N,N}^2(u)}{2\sqrt{t}}, \\ \tilde{\beta}_{N,N}(t) &= (3u)^{1/3} \left(\beta_{N,N}(u) - \frac{1}{6u} \right) = \frac{1}{(4t)^{1/4}} \left(\beta_{N,N}(u) - \frac{1}{6u} \right). \end{aligned} \quad (9.5)$$

We have the following relation between the free energy in the variables u and t :

$$\tilde{F}_N(t) = \frac{1}{108u^2} + \frac{\ln(3u)}{3} + F_N(u) = \frac{2t^{3/2}}{3} - \frac{\ln(4t)}{4} + F_N(u). \quad (9.6)$$

We start from the regular regime, which can be extended to cover the range $u_c - u = \mathcal{O}(N^{-4/5+\varepsilon})$. In this case

$$\tilde{F}_N(t) = \tilde{F}_0(t) + \frac{\tilde{F}_2(t)}{N^2} + \mathcal{O}(N^{-4}). \quad (9.7)$$

The behavior of $\tilde{F}_N(t)$ near the critical point can be analyzed as in [7], using the Toda equation (9.1), bearing in mind that the coefficient $\tilde{\gamma}_{N,N}^2(t)$ admits an asymptotic expansion in powers of N^{-2} if $t > t_c$:

$$\tilde{\gamma}_{N,N}^2(t) = \tilde{g}_0(t) + \frac{\tilde{g}_2(t)}{N^2} + \mathcal{O}(N^{-4}). \quad (9.8)$$

The first term $\tilde{g}_0(t)$ behaves as follows near $t = t_c$:

$$\tilde{g}_0(t) = 2^{-2/3} - 2^{-1/3} 3^{-1/2} (\Delta t)^{1/2} + \mathcal{O}(\Delta t), \quad \Delta t = t - t_c. \quad (9.9)$$

Integrating twice (9.1) from $t = t_c$, we obtain

$$\tilde{F}^{(0)}(t) = -\frac{2^{5/3} 3^{1/2}}{45} (\Delta t)^{5/2} + 2^{-5/3} (\Delta t)^2 + \tilde{A} + \tilde{B} \cdot \Delta t + \mathcal{O}((\Delta t)^3), \quad (9.10)$$

where \tilde{A} and \tilde{B} are constants. The next term is

$$\tilde{F}^{(2)}(t) = -\frac{1}{48} \ln(\Delta t) + \tilde{D} + \mathcal{O}((\Delta t)^{1/2}), \quad (9.11)$$

where \tilde{D} is a constant. We add and subtract the logarithmic term, which is singular at $t = t_c$:

$$\tilde{F}_N(t) = \tilde{F}^{(0)}(t) + \frac{\tilde{F}^{(2)}(t) + \frac{1}{48} \ln(\Delta t)}{N^2} - \frac{\ln(\Delta t)}{48N^2} + \mathcal{O}(N^{-4}) \quad (9.12)$$

We now define

$$\tilde{F}_N^{\text{reg}}(t) = 2^{-5/3} (\Delta t)^2 + \tilde{A} + \tilde{B} \cdot \Delta t + \frac{\tilde{D}}{N^2} \quad (9.13)$$

Note that Δt and Δu are essentially equivalent (up to a constant) near the critical value, so

$$F_N^{\text{reg}}(u) = A + B\Delta u + C(\Delta u)^2 + \frac{D}{N^2}, \quad (9.14)$$

for some constants A , B , C and D , that come both from the change of variables from t to u and the prefactor in (9.6).

The function $F^{(0)}(u)$ can actually be written in terms of generalized hypergeometric functions, with the aid of MAPLE:

$$F^{(0)}(u) = 6u^2 + 216u^4 {}_4F_3 \left(\begin{matrix} 1 & 1 & \frac{4}{3} & \frac{5}{3} \\ 2 & \frac{5}{2} & 3 \end{matrix} ; 34992u^4 \right) + 13608u^6 {}_4F_3 \left(\begin{matrix} 1 & \frac{3}{2} & \frac{11}{6} & \frac{13}{6} \\ 3 & \frac{5}{2} & \frac{7}{2} \end{matrix} ; 34992u^4 \right) \quad (9.15)$$

This representation is not particularly useful for computations, but it allows us to determine the regularity of the function $F^{(0)}(u)$ at the critical point. Using standard arguments, see for instance [1, §16.2], this function is absolutely convergent when $34992u^4 = 1$, which corresponds in particular to $u = u_c$. Then we have

$$A = F^{(0)}(u_c) = 54\sqrt{3} + \frac{1}{162} {}_4F_3 \left(\begin{matrix} 1 & 1 & \frac{4}{3} & \frac{5}{3} \\ 2 & \frac{5}{2} & 3 \end{matrix} ; 1 \right) + \frac{7\sqrt{3}}{5832} {}_4F_3 \left(\begin{matrix} 1 & \frac{3}{2} & \frac{11}{6} & \frac{13}{6} \\ 3 & \frac{5}{2} & \frac{7}{2} \end{matrix} ; 1 \right). \quad (9.16)$$

Differentiation with respect to u increases each parameter of the hypergeometric functions by one, and it is not difficult to check that $F^{(0)}(u)$ is twice differentiable at $u = u_c$. Therefore, we get

$$B = F^{(0)'}(u_c), \quad C = \frac{1}{2} F^{(0)''}(u_c). \quad (9.17)$$

The non-analytic terms in $\tilde{F}_N(t)$ can be determined from the double scaling regime. Consider the Toda equation (9.1), which is valid for any $t > t_c$, see [7]. We can establish a connection between the variables λ and t , from the double scaling setting and the change of variables (9.2). Namely, if $\Delta u = u_c - u$ and $\Delta t = t - t_c$, we have

$$\Delta u = 2^{-7/3} 3^{-7/4} \Delta t + \mathcal{O}((\Delta t)^2) \quad (9.18)$$

and since $\lambda = -2^{12/5} 3^{7/4} N^{4/5} \Delta u$, from the double scaling relation, we get

$$\lambda = -2^{1/15} N^{4/5} \Delta t + \mathcal{O}(N^{-4/5}). \quad (9.19)$$

We define the auxiliary variable

$$\nu = -2^{1/15} N^{4/5} \Delta t, \quad (9.20)$$

and observe that $\nu - \lambda = \mathcal{O}(N^{-4/5})$. The Toda equation (9.1) can be written in terms of this variable ν :

$$\frac{d^2 \tilde{F}_N(\nu)}{d\nu^2} = 2^{-2/15} N^{-8/5} \tilde{\gamma}_{N,N}^2(\nu), \quad (9.21)$$

and then, since

$$\tilde{\gamma}_{N,N}^2(t) = \frac{1}{2\sqrt{t}} \gamma_{N,N}^2(u), \quad (9.22)$$

we have

$$\tilde{\gamma}_{N,N}^2(\nu) = \frac{1}{2\sqrt{t_c}} \gamma_{N,N}^2(\nu) + \mathcal{O}(N^{-4/5}) = 2^{-2/3} 3^{-1/2} \gamma_{N,N}^2(\nu) + \mathcal{O}(N^{-4/5}). \quad (9.23)$$

Then, substituting the asymptotic expansion for $\gamma_{N,N}^2(\nu)$ in (9.21), we get the following differential equation:

$$\begin{aligned} \frac{d^2 \tilde{F}_N(\nu)}{d\nu^2} &= \frac{2^{-4/5} 3^{-1/2}}{N^{8/5}} \gamma_{N,N}^2(\nu) + \mathcal{O}(N^{-12/5}) \\ &= \frac{2^{-4/5}}{N^{8/5}} - \frac{y_\alpha(\nu)}{N^2} + \mathcal{O}(N^{-12/5}). \end{aligned} \quad (9.24)$$

Integrating twice in ν , we have

$$\tilde{F}_N(\nu) = \frac{2^{-9/5} \nu^2}{N^{8/5}} + A_N \nu + B_N - \frac{Y_\alpha(\nu)}{N^2} + \mathcal{O}(N^{-12/5}). \quad (9.25)$$

Here A_N and B_N are constants of integration. In terms of Δt , using (9.20), we have

$$\begin{aligned}\tilde{F}_N(t) &= 2^{-5/3}(\Delta t)^2 - A_N 2^{1/15} N^{4/5} \Delta t + B_N - \frac{Y_\alpha(\nu)}{N^2} + \mathcal{O}(N^{-12/5}), \\ &= \tilde{F}_N^{\text{reg}}(t) + \tilde{F}_N^{\text{sing}}(\nu) + \mathcal{O}(N^{-12/5}),\end{aligned}\tag{9.26}$$

where the term $\tilde{F}_N^{\text{reg}}(t)$ matches the one before, and

$$\tilde{F}_N^{\text{sing}}(\nu) = -\frac{Y_\alpha(\nu)}{N^2}.\tag{9.27}$$

This function $Y_\alpha(\nu)$ satisfies the ODE

$$Y_\alpha''(\nu) = y_\alpha(\nu),\tag{9.28}$$

with boundary condition

$$Y_\alpha(\nu) = \frac{2\sqrt{6}}{45}(-\nu)^{5/2} - \frac{1}{48} \ln(-\nu) + \mathcal{O}(\nu^{-5/2}), \quad (-\nu) \rightarrow \infty.\tag{9.29}$$

This boundary condition is chosen to match the two non-analytic terms at $t = t_c$ appearing in the regular regime. Finally, we use the fact that $\nu = \lambda + \mathcal{O}(N^{-4/5})$, and define

$$F_N^{\text{sing}}(\lambda) = -\frac{Y_\alpha(\lambda)}{N^2},\tag{9.30}$$

which completes the proof of the theorem.

APPENDIX A. PROOF OF THEOREM 3.2

We start with the Riemann–Hilbert problem for the function $\Phi(\zeta; \lambda, \alpha)$. First of all, we translate the contour Γ_Ψ to $\Gamma_\Phi = \Gamma_\Psi + \zeta_0$, where $\zeta_0 = -2/\sqrt{6}$, recall (3.39). This implies a modification of Φ in different sectors taking into account the jumps on the contour Γ_Ψ , but it does not alter the asymptotic behavior. Thus, we consider the following RH problem for $\Phi(\zeta; \lambda, \alpha)$:

- (1) Φ is analytic on $\mathbb{C} \setminus \Gamma_\Phi$, where the contour Γ_Φ is depicted in Figure 3, but centered at the point $\zeta = \zeta_0$.
- (2) On Γ_Φ , Φ has the following jumps:

$$\Phi_+(s) = \Phi_-(s) \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & s \in \gamma_{\pm 2} \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & s \in \rho \\ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, & s \in \gamma_1 \\ \begin{pmatrix} 1 & 1 - \alpha \\ 0 & 1 \end{pmatrix}, & s \in \gamma_{-1}. \end{cases}\tag{A.1}$$

- (3) As $\zeta \rightarrow \infty$, for fixed λ and α , Φ expands in the asymptotic series

$$\Phi(\zeta; \lambda, \alpha) \sim \frac{(\zeta - \zeta_0)^{\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \left(I + \sum_{k=1}^{\infty} \frac{\Phi_k(\lambda, \alpha)}{(-\lambda)^{k/4} \zeta^{k/2}} \right) e^{(-\lambda)^{5/4} g(\zeta) \sigma_3},\tag{A.2}$$

where we recall that

$$g(\zeta) = \frac{4}{5} \left(\zeta + \frac{2}{\sqrt{6}} \right)^{5/2} - \frac{2\sqrt{6}}{3} \left(\zeta + \frac{2}{\sqrt{6}} \right)^{3/2},\tag{A.3}$$

with a cut on the axis $(-\infty, \zeta_0]$.

A.1. First transformation. Consider the new matrix function

$$S(\zeta; \lambda, \alpha) = \Phi(\zeta; \lambda, \alpha) e^{-(-\lambda)^{5/4} g(\zeta) \sigma_3}, \quad (\text{A.4})$$

Then $S(\zeta; \lambda, \alpha)$ satisfies the following RH problem:

- (1) S is analytic on $\mathbb{C} \setminus \Gamma_\Phi$.
- (2) On Γ_Φ , S has the following jumps:

$$S_+(s) = S_-(s) \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{-2(-\lambda)^{5/4} g(s)} & 1 \end{pmatrix}, & s \in \gamma_{\pm 2} \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & s \in \rho \\ \begin{pmatrix} 1 & \alpha e^{2(-\lambda)^{5/4} g(s)} \\ 0 & 1 \end{pmatrix}, & s \in \gamma_1 \\ \begin{pmatrix} 1 & (1-\alpha) e^{2(-\lambda)^{5/4} g(s)} \\ 0 & 1 \end{pmatrix}, & s \in \gamma_{-1}. \end{cases} \quad (\text{A.5})$$

- (3) As $\zeta \rightarrow \infty$, S expands in the asymptotic series

$$S(\zeta; \lambda, \alpha) \sim \frac{(\zeta - \zeta_0)^{\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \left(I + \sum_{k=1}^{\infty} \frac{\Phi_k(\lambda, \alpha)}{(-\lambda)^{k/4} \zeta^{k/2}} \right) \quad (\text{A.6})$$

Note that the jump on ρ is the same as before, since it is not difficult to check that the function $g(\zeta)$ has a jump on $(-\infty, \zeta_0]$, but it is true that $g_+(s) = -g_-(s)$ for $s \in (-\infty, \zeta_0]$.

Let us deform the contour slightly near the point $\zeta = \zeta_0$, as indicated in Figure 11. It is not difficult to check that all the jumps remain the same, except a new one on γ_0 , which is equal to

$$S_+(s) = S_-(s) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad s \in \gamma_0. \quad (\text{A.7})$$

Taking into account the level curves corresponding to $\operatorname{Re} g(\zeta) = 0$, see Figure 11, it is clear that the jumps on $\gamma_{\pm 1}$ and on $\gamma_{\pm 2}$ tend to the identity exponentially fast, provided that the segment γ_0 does not enter the region where $\operatorname{Re} g(\zeta) < 0$.

A.2. Model RH problem. Now we ignore all the jumps exponentially close to identity (that is, the ones on $\gamma_{\pm 1}$ and on $\gamma_{\pm 2}$), and we solve the model Riemann–Hilbert problem: we look for $M(\zeta)$ such that

- (1) M is analytic on $\mathbb{C} \setminus (-\infty, \zeta_0]$.
- (2) On $(-\infty, \zeta_0)$, we have

$$M_+(s) = M_-(s) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (\text{A.8})$$

- (3) As $\zeta \rightarrow \infty$, $\zeta \notin (-\infty, \zeta_0]$, we have

$$M(\zeta) = \frac{(\zeta - \zeta_0)^{\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} (I + \mathcal{O}(\zeta^{-1})). \quad (\text{A.9})$$

This model RH problem can be solved explicitly, namely

$$M(\zeta) = \frac{(\zeta - \zeta_0)^{\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}. \quad (\text{A.10})$$

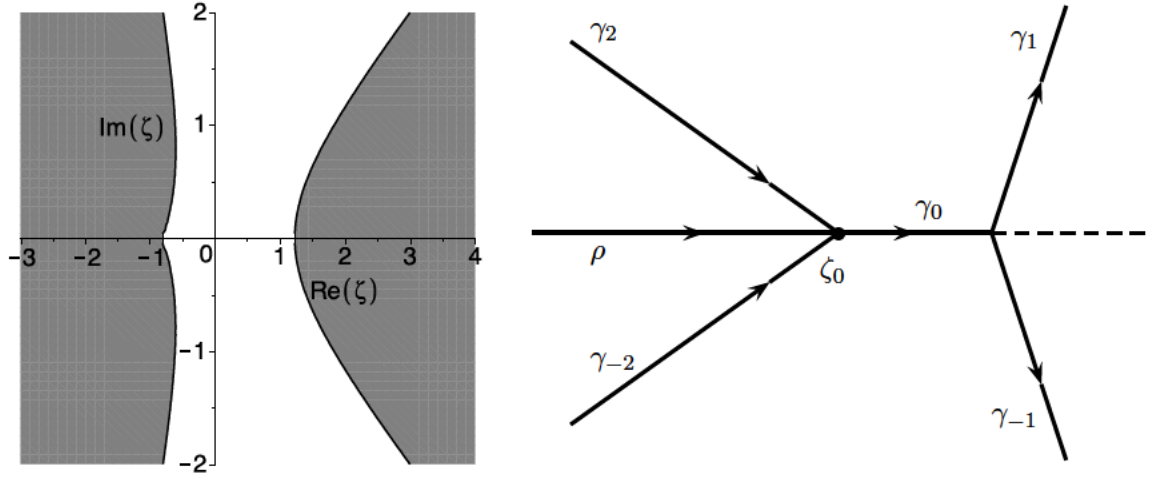


FIGURE 11. On the left, level curves where $\operatorname{Re} g(\zeta) = 0$. In grey, the region where $\operatorname{Re} g(\zeta) > 0$, in white where $\operatorname{Re} g(\zeta) < 0$. On the right, the modified contour Γ_S .

A.3. Local parametrix around $\zeta = \zeta_0$. Take a fixed disc $D(\zeta_0, \varepsilon)$, and consider the following RH problem for a function $P(\zeta; \lambda, \alpha)$:

- (1) P is analytic on $D(\zeta_0, \varepsilon) \cap \Gamma_S$.
- (2) In $D(\zeta_0, \varepsilon) \cap \Gamma_S$, P has the following jumps:

$$P_+(s) = P_-(s) \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{-2(-\lambda)^{5/4}g(s)} & 1 \end{pmatrix}, & s \in \gamma_{\pm 2} \cap D(\zeta_0, \varepsilon) \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & s \in \rho \cap D(\zeta_0, \varepsilon) \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & s \in \gamma_0 \cap D(\zeta_0, \varepsilon) \end{cases} \quad (\text{A.11})$$

- (3) Uniformly for $\zeta \in \partial D(\zeta_0, \varepsilon)$, we have the matching

$$P(\zeta; \lambda, \alpha) = M(\zeta) \left(I + \mathcal{O}((- \lambda)^{5/4}) \right). \quad (\text{A.12})$$

The reduction to constant jumps is standard now: consider

$$\tilde{P}(\zeta; \lambda, \alpha) = P(\zeta; \lambda, \alpha) e^{(-\lambda)^{5/4}g(\zeta)\sigma_3}, \quad (\text{A.13})$$

then $\tilde{P}(\zeta; \lambda, \alpha)$ satisfies the following RH problem:

- (1) \tilde{P} is analytic on $D(\zeta_0, \varepsilon) \cap \Gamma_S$.
- (2) $D(\zeta_0, \varepsilon) \cap \Gamma_S$, \tilde{P} has the following jumps:

$$\tilde{P}_+(s) = \tilde{P}_-(s) \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & s \in \gamma_{\pm 2} \cap D(\zeta_0, \varepsilon) \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & s \in \rho \cap D(\zeta_0, \varepsilon) \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & s \in \gamma_0 \cap D(\zeta_0, \varepsilon) \end{cases} \quad (\text{A.14})$$

(3) Uniformly for $\zeta \in \partial D(\zeta_0, \varepsilon)$, we have the matching

$$\tilde{P}(\zeta; \lambda, \alpha) = M(\zeta) \left(I + \mathcal{O}((- \lambda)^{-5/4}) \right) e^{(-\lambda)^{5/4} g(\zeta) \sigma_3}. \quad (\text{A.15})$$

This is a standard Airy Riemann–Hilbert problem, so we look for a local parametrix in the form

$$\tilde{P}(\zeta; \lambda, \alpha) = E(\zeta; \lambda) A((-\lambda)^{5/6} f(\zeta)), \quad (\text{A.16})$$

where $E(\zeta; \lambda)$ is an analytic prefactor, the matrix $A(w)$ is built of suitably chosen Airy functions, and $f(\zeta)$ is a conformal mapping from a neighborhood of ζ_0 onto a neighborhood of 0 in the auxiliary w complex plane. The important fact for us in this case is that $A((-\lambda)^{5/6} f(\zeta))$ can be expanded asymptotically as the argument grows large:

$$A((-\lambda)^{5/6} f(\zeta)) = ((-\lambda)^{5/6} f(\zeta))^{-\sigma_3/4} \frac{1}{2\sqrt{\pi}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left(I + \mathcal{O}((- \lambda)^{-5/4}) \right) e^{-\frac{2}{3}(-\lambda)^{5/4} (f(\zeta))^{3/2} \sigma_3}. \quad (\text{A.17})$$

Now we choose $f(\zeta)$ in the following way:

$$f(\zeta) = \left(-\frac{3}{4} g(\zeta) \right)^{3/2}, \quad (\text{A.18})$$

to match the exponential factors, and since from (3.34) we can write

$$g(\zeta) = -\frac{2\sqrt{6}}{3} (\zeta - \zeta_0)^{3/2} \left(1 - \frac{\sqrt{6}}{5} (\zeta - \zeta_0) \right), \quad (\text{A.19})$$

then it follows that $f(\zeta)$ is indeed a conformal mapping in a neighborhood of $\zeta = \zeta_0$. Finally, to get the correct matching with $M(\zeta)$ in (A.15), we take

$$E(\zeta; \lambda) = \sqrt{\pi} M(\zeta) \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} ((-\lambda)^{5/6} f(\zeta))^{-\sigma_3/4}, \quad (\text{A.20})$$

which is analytic for $\zeta \in (-\infty, \zeta_0]$ because of the jumps of $M(\zeta)$ and the fractional power $f(\zeta)^{-\sigma_3/4}$.

A.4. Second transformation. Finally, we define

$$R(\zeta; \lambda, \alpha) = S(\zeta; \lambda, \alpha) \begin{cases} M^{-1}(\zeta), & \zeta \in \mathbb{C} \setminus D(\zeta_0, \varepsilon), \\ P^{-1}(\zeta), & \zeta \in D(\zeta_0, \varepsilon). \end{cases} \quad (\text{A.21})$$

Then this matrix $R(\zeta; \lambda, \alpha)$ is analytic in $\mathbb{C} \setminus \Gamma_R$, where Γ_R is depicted in Figure 12. It has exponentially small jumps in $(-\lambda)$ in the whole contour except on the boundary of the disc, $\partial D(\zeta_0, \varepsilon)$ where it is of order $(-\lambda)^{5/4}$.

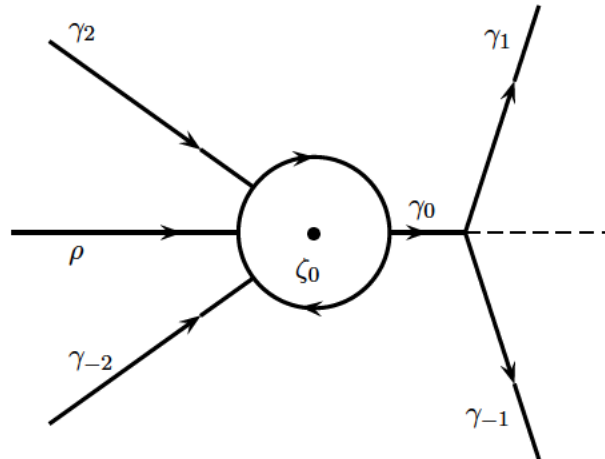
Following standard arguments, see for instance [6], we can conclude that as $\lambda \rightarrow -\infty$, we have

$$R(\zeta; \lambda, \alpha) = I + \mathcal{O} \left(\frac{1}{(-\lambda)^{5/4} (1 + |\zeta|)} \right), \quad (\text{A.22})$$

uniformly for $\zeta \in \mathbb{C} \setminus D(\zeta_0, \varepsilon)$. Now, undoing the transformations, we get

$$\begin{aligned} \Phi(\zeta; \lambda, \alpha) &= S(\zeta; \lambda, \alpha) e^{(-\lambda)^{5/4} g(\zeta) \sigma_3} \\ &= R(\zeta; \lambda, \alpha) M(\zeta) e^{(-\lambda)^{5/4} g(\zeta) \sigma_3} \\ &= \left(I + \mathcal{O} \left(\frac{1}{(-\lambda)^{5/4} (1 + |\zeta|)} \right) \right) M(\zeta) e^{(-\lambda)^{5/4} g(\zeta) \sigma_3}, \end{aligned} \quad (\text{A.23})$$

valid as $(-\lambda) \rightarrow \infty$, uniformly for $\zeta \in \mathbb{C} \setminus D(\zeta_0, \varepsilon)$. This completes the proof of the theorem.


 FIGURE 12. The final contour Γ_R .

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